SPECIAL VALUED ℓ-GROUPS

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Tech Report #CS84011-R

10/24/84

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Abstract

Special elements and values have always been of interest in the study of lattice-ordered groups, arising naturally from totally-ordered groups and lexicographic extensions. Much work has been done recently with the class of lattice-ordered groups whose root system of regular subgroups has a plenary subset of special values. We call such l-groups *special valued.*

In this paper, we first show that several familiar structures of l-groups, namely polars, minimal prime subgroups, and the lex kernel, are recognizable from the lattice and the identity; that is, knowing which element of the lattice is the group identity, we can pick out in the lattice all the elements of polars, minimal primes, and the lex kernel. This then leads to an easy proof that special elements can be recognized from the lattice and the identity.

We then prove several results about the class S of special-valued l-groups. We give a simple and direct proof that S is closed with respect to joins of convex l-subgroups, incidentally giving a direct proof that S is a quasitorsion class. This proof is then used to show that the special-valued and finite-valued kernels of l-groups are recognizable from the lattice and the identity. We show also that the lateral completion of a special-valued l-group is special-valued and is an a^*-extension of the original l-group.

Our most important result is that the lateral completion of a completely-distributive normal-valued l-group is special-valued. This lends itself easily to a new and simpler proof of Ball, Conrad, and Darnel's result that every normal-valued l-group can be l-embedded into a special-valued l-group. Readers familiar with the impact of the Conrad-Harvey-Holland Theorem on abelian l-groups will recognize the importance of the last theorem to the study of the class of normal-valued l-groups and to the study of proper varieties of l-groups, all of which are normal valued.

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SECTION 1: Introduction

In this section, we gather many of the terms and fundamental results about l-groups that we will use throughout the paper.

We will follow tradition and use additive notation for the group operation throughout the paper even though most of the groups will not be abelian. $\Re$ will denote the group of real numbers, $\mathcal{Q}$ the group of rational numbers,and Z the group of integers, ordered in the usual way.

A *lattice* is a partially-ordered set in which every pair of elements has a least upper bound and a greatest lower bound. A *lattice-ordered group*, written l-group, is a group whose underlying set is lattice-ordered and such that if x, y, g, and h are elements of the group $g \leq h$, then $x+g+y \leq x+h+y$. If the order is a total order, the group is called an o-group.

If G is an l-group and $g \in G$, the *positive part* of g, written g^+, is $g \vee 0$, and the *negative part* of g, written g^-, is $-g \vee 0$. $g^+ \wedge g^- = 0$ and $g = g^+ - g^-$; in fact, this is the unique such representation of g: if $h \wedge k = 0$ and $g = h - k$, then $h = g^+$ and $k = g^-$. The *absolute value* of g, written $|g|$, is $g^+ + g^- = g^+ \vee g^- = g \vee -g$. Two elements x and y of G are *disjoint* if $|x| \wedge |y| = 0$. x is a *component* of g if $|x| \wedge |g - x| = 0$. An element a is *infinitely smaller* than an element b if $na \leq b$ for every integer n. This is denoted by $a << b$. An l-group is *archimedean* if $0 \leq a << b$ forces $a = 0$. A central theorem of l-groups is Hölder's Theorem [ ] that an archimedean o-group is isomorphic, as an ordered group, to a subgroup of $\Re$.

A subgroup A of G is an *l-subgroup* if A is also a sublattice of G; this is equivalent to $a \in A$ forcing $a \vee 0$ to also be an element of A. An l-subgroup A is *dense* in G if for any $0 < g$ in G, there exists $a \in A$ such that $0 < a \leq g$. An l-subgroup A is *convex* if $0 \leq g \leq a \in A$ forces $g \in A$. The set of convext l-subgroups of G, denoted $\mathcal{C}(G)$, is a sublattice of the lattice of all subgroups of G. A normal convex l-subgroup is called an *l-ideal*; $\mathcal{L}(G)$ denotes the set of l-ideals of G and is a sublattice of $\mathcal{C}(G)$.

If both G and H are l-groups and τ: $G \to H$ is both a group and a lattice homomorphism, τ is called an *l-homomorphism*. A group homomorphism τ is an l-homomorphism if and only if for every $g \in G$, $\tau(g \vee 0) = \tau(g) \vee 0$. An l-homomorphism τ is *complete* if τ preserves all meets and joins. The kernel of an l-homomorphism is always an l-ideal and the natural homomorphism of an l-group onto the factor group of an l-ideal is always an l-homomorphism.

A convex l-subgroup P is *prime* if $a \wedge b = 0$ forces either $a \in P$ or $b \in P$. The set of prime subgroups of an l-group forms a *root system* under inclusion: no two incomparable elements have a lower bound. The intersection of a chain of prime subgroups is prime and thus, by Zorn's Lemma, every prime subgroup contains at least one minimal prime subgroup.

For any $0 \neq g \in G$, there exists at least one convex l-subgroup M that is maximal with respect to not containing g. M is called a *regular* subgroup and is a *value* of g. Regular subgroups are prime and also form a root system under inclusion, commonly denoted $\Gamma(G)$. If M is a regular subgroup, M is properly contained in $M^* = \cap\{N \in \mathcal{C}(G) : M \subset N\}$. M^* is called the *cover* of M. It is customary to use $\Gamma(G)$ as an index set of small Greek letters and, if $\delta \in \Gamma(G)$, to let G_δ denote the associated regular subgroup and G^δ the cover of G_δ. A *plenary subset* of $\Gamma(G)$ is a dual ideal whose intersection (of the corresponding regular subgroups) is the identity. We will call a plenary subset Δ *normal* if for every $G_\delta \in \Delta$ and any $g \in G$, $-g + G_\delta + g$ is also in Δ. A regular subgroup M is a *normal value* if M is a normal subgroup of M^*, in which case M^*/M is an archimedean o-group. A *normal-valued l-group* is an l-group in which every value is a normal value; this is equivalent to a plenary subset of $\Gamma(G)$ consisting of normal values.

For any subset S of G, the *polar* of S, denoted S', is the set $\{g \in G : |g| \wedge |s| = 0$ for all $s \in S\}$. Polars are convex l-subgroups of G and form a complete Boolean algebra under inclusion. If $\{A_\lambda\}_{\lambda\in\Lambda}$ is a set of l-groups, then on the group direct product of the set $\{A_\lambda\}_{\lambda\in\Lambda}$, we place an order by defining $(..., a_\lambda, ...) \geq (..., 0, ...)$ if for every $\lambda \in \Lambda$, $a_\lambda \geq 0$. The resulting l-group is called the *cardinal product* of the set $\{A_\lambda\}_{\lambda\in\Lambda}$ and is denoted $\Pi_\Lambda A_\lambda$. An important convex l-subgroup of $\Pi_\Lambda A_\lambda$ is the *cardinal sum* ΣA_λ, consisting of those elements a such that for all but a finite number of $\lambda \in \Lambda$, $a_\lambda = 0$.

A convex l-subgroup A is *closed* if whenever $\{a_\lambda\}_{\lambda\in\Lambda}$ is a subset of A and $\bigvee_{\lambda\in\Lambda} a_\lambda$ exists in G, then $\bigvee_{\lambda\in\Lambda} a_\lambda$ is in A. Polars are known to be closed convex l-subgroups; $\mathcal{K}(G)$ will denote the set of closed convex l-subgroups of the l-group G. $\mathcal{K}(G)$ does form a lattice under inclusion but need not be a sublattice of $\mathcal{C}(G)$. An l-ideal is closed if and only if the natural homomorphism onto the factor group is complete.

An l-group G is *completely distributive* if whenever $\{g_{ij}\}_{i\in I, j\in J}$ is a subset of G and both $\bigvee_I \bigwedge_J g_{ij}$ and $\bigwedge_{I_J} \bigvee_I g_{i,f(i)}$ exist, they must be equal. A normal-valued l-group is completely-distributive if and only if $\Gamma(G)$ has a necessarily unique minimal plenary subset of closed values; we will denote this minimal plenary subset of $\Gamma(G)$ by $\Delta(G)$.

All of the above results can be found either in [10] or in [17]. We will also need the following definitions and results which are also widely known but which, by and large, have come after the above texts were written.

An element g is *special* if g has only one value, in which case its value is called special as well. A special value is necessarily a normal value [17]. If an element has only a finite number of values, each value must be special [14]. An l-group is called *finite-valued* if every nonzero element has a finite number of values. If any element g has a special value M, then g has a special component with value M [15]. We introduce and use the term *special-valued* to denote an l-group whose root system of regular subgroups has a (necessarily minimal) plenary subset of special values. A special-valued l-group is thus a completely-distributive normal-valued l-group. The most fundamental result of special-valued l-groups is that an l-group is special-valued if and only if every positive element is the join of a pairwise disjoint collection of positive special elements [15].

If Δ is a root system, let $V(\Delta, \Re)$ denote the functions from Δ to $\Re$ whose supports satisfy the ascending chain condition on the root system Δ. $V(\Delta, \Re)$ is a group under component-wise addition and, defining $f \in V(\Delta, \Re)$ to be positive if $f(\delta) > 0$ for every maximal component δ in the support of f, $V(\Delta, \Re)$ is then a special-valued l-group, where the special values are all of the form (for $\delta \in \Delta$) $V_\delta = \{f \in V(\Delta, \Re) : f(\alpha) = 0 \text{ for all } \alpha \geq \delta\}$. If G is an abelian l-group and Δ is a plenary subset of $\Gamma(G)$, there exists an l-homomorphism τ of G into $V(\Delta, \Re)$ such that $g \in G^\delta \setminus G_\delta$ if and only if $\tau(g) \in V^\delta \setminus V_\delta$ for all $\delta \in \Delta$ [19]. This is of course the most quoted theorem in l-group theory, the Conrad-Harvey-Holland Theorem.

Finally, an *l-variety* is an equationally-defined class of l-groups. The class $\mathcal{N}$ of normal-valued l-groups is such a variety, satisfying the equation $[(a\vee 0)+(b\vee 0)]\wedge[2(b\vee 0)+2(a\vee 0)] = (a\vee 0)+(b\vee 0)$ [10]. A *torsion class* is a class $\mathcal{T}$ of l-groups closed under convex l-subgroups, l-homomorphic images, and joins of convex l-subgroups [26]. The class $\mathcal{F}$ of finite-valued l-groups is a torsion class [26] and every l-variety is a torsion class [22]. A *quasitorsion class* is a class of l-groups that differs from being a torsion class in that it need contain only complete l-homomorphic images and not every l-homomorphic image [24]. The class of archimedean l-groups is a quasitorsion class [24] as is the class $\mathcal{S}$ of special-valued l-groups [18]. Clearly every torsion class is a quasitorsion class. The final generalization of a variety is a *radical class*, which is closed with respect to convex l-subgroups, l-isomorphic images, and joins of convex l-subgroups [23]. Clearly a quasitorsion class is a radical class and examples of radical classes that are not quasitorsion classes can be found in [18] and [23].

If $\mathcal{R}$ is a radical class of l-groups, let α be a successor ordinal and suppose that $\mathcal{R}^{\alpha-1}$ is defined. We define $\mathcal{R}^{\alpha}$ to be the class of l-groups G that contain an l-ideal A such that A is in $\mathcal{R}^{\alpha-1}$ and G/A is in $\mathcal{R}$. If α is a limit ordinal, we define $\mathcal{R}^{\alpha}$ to the class of l-groups G such that there exists a set $\{A_\beta\}_{\beta<\alpha}$ of l-ideals of G such that $A_\beta \in \mathcal{R}_\beta$, $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$, and $G = \bigcup_{\beta<\alpha} A_\beta$. Then for any ordinal α, $\mathcal{R}^{\alpha}$ is a radical class [23] and is a quasitorsion class [24] or a torsion class [26] if $\mathcal{R}$ is. If $\mathcal{V}$ is an l-variety and n is a positive integer, $\mathcal{V}^n$ is also an l-variety [25].

SECTION 2: Special Elements

The purpose of this section is to show that special elements can be recognized in the underlying lattice of an l-group. This statement needs some explanation because, since group automorphisms are always lattice automorphisms, no point in the underlying lattice can be distinguished from any other point.

So what exactly is meant when we say that polars, minimal prime subgroups, and special elements can be recognized in the lattice? Basically this: if $+$ and $\oplus$ are two group operations with the same identity on the lattice $(G, \leq)$ such that $(G, \leq, +)$ and $G, \leq, \oplus)$ are both l-groups, then a subset of G is a polar of $(G, \leq, +)$ if and only if it is also a polar of $(G, \leq, \oplus)$. We do need that $+$ and $\oplus$ have the same identity, but this is not as much a restriction as it might seem. For if θ_1 and θ_2 are the respective identities for $+$ and $\oplus$, let τ denote the map $g \rightarrow g + \theta_2$. Defining a third group operation $*$ on G by $g * h = \tau^{-1}[\tau(g) \oplus \tau(h)]$ yields that θ_1 is the identity of $*$ and that $(G, \leq, *)$ is an l-group isomorphic to $(G, \leq, \oplus)$. Thus it does no harm to assume that $+$ and $\oplus$ have the same identity. We will call any group operation, $+$, on the lattice $(G, \leq)$ such that $(G, \leq, +)$ is an l-group an *l-operation.*

Now if $+$ and $\oplus$ are l-operations on $(G, \leq)$, the map $g \rightarrow - \ominus g$ is easily seen to be a lattice automorphism of $(G, \leq)$. This fact underlies most of the known results about changing the group operation on a lattice.

Lemma 2.1. Let $+$ and $\oplus$ be l-operations on $(G, \leq)$ and let C be a convex submonoid of G^+ under both l-operations such that the map $g \rightarrow - \ominus g$ is a permutation of C. Then the convex l-subgroups $[C, +]$ and $[C, \oplus]$ of $(G, +)$ and $(G, \oplus)$, respectively, generated by C are the same subset of G.

PROOF: Since C is a convex submonoid under both $+$ and $\oplus$, $C = [C, +]^+ = [C, \oplus]^+$. If $g \in [C, \oplus]^-$, $-g \in C$ implies that $-g = - \ominus c$ for some $c \in C$ and so $g = \ominus c$, placing g in $[C, +]^-$. Thus $[C, +]^- \subseteq [C, \oplus]^-$ and a symmetric argument shows that $[C, \oplus]^- \subseteq [C, +]^-$.

Now let $g \in [C, +]$ which of course implies that $g = c_1 - c_2$, where c_1 and $c_2 \in C$ and $c_1 \wedge c_2 = 0$. Then $-c_1 - c_2 \leq c_1 - c_2 \leq c_1 + c_2$. Since both $-c_1 - c_2$ and $c_1 + c_2$ are in $[C, \oplus]$

and $[C,\oplus]$ is convex, $c_1 - c_2$ is in $[C,\oplus]$. Thus $[C,+] \subseteq [C,\oplus]$ and symmetry gives equality of the sets.

Lemma 2.2. (due to Paul Conrad) Let $+$ and $\oplus$ be l-operations on $(G,\leq)$. Then for any $g > 0$ in G, $g \wedge - \ominus g > 0$ and $g \wedge \ominus - g > 0$.

PROOF: Suppose $g \wedge - \ominus g = 0$. Then $g > g + \ominus g > \ominus g$ implies that $g + \ominus g \in (G(g),\oplus)$, the convex l-subgroup of $(G,\oplus)$ that is generated by g. Thus $g + \ominus g = h_1 \ominus h_2$, where h_1 and h_2 are positive elements of $(G(g),\oplus)$ and $h_1 \wedge h_2 = 0$. But $h_1 = (g + \ominus g) \vee 0 = g$ and so $h_2 \wedge g = 0$, a contradiction.

These two lemmas now give us a theorem that is and is not surprising, namely that polars are lattice-recognizable. This is not surprising in one sense for the positive cone of a polar is of course defined in terms of the lattice and the group identity. But what is surprising is that the negative cone is determined by the lattice and the identity as well as all elements that are incomparable to the identity.

Theorem 2.3. Polars of an l-group are recognizable from the lattice and the identity.

PROOF: Let $+$ and $\oplus$ be l-operations on $(G,\leq)$, $0 < g \in G$ and $0 < h \in g'$. Assume $- \ominus g \wedge h > 0$; then $g \wedge \ominus - h > 0$. But then we have that $0 = g \wedge h \geq g \wedge \ominus - h \wedge - \ominus g \wedge h = (g \wedge \ominus - h) \wedge - \ominus (g \wedge \ominus - h) > 0$, a contradiction. So $- \ominus g \wedge h = 0$ implying that $- \ominus g$ and $\ominus - g$ are both elements of g''. Thus if P is a polar of $(g,+)$ and $0 < g \in P$, both $- \ominus g$ and $\ominus - g \in P$ implying, of course, that $-\ominus$ is a permutation of $P+$ which is a convex submonoid of both $(G,+)$ and $(G,\oplus)$.

Corollary 2.4. Minimal prime subgroups are recognizable from the lattice and the identity.

PROOF: M is a minimal prime subgroup of $(G,+)$ if and only if $M = \cup\{x' : x \notin M\}$ which is true if and only if M is a minimal prime subgroup of $(G,\oplus)$ [17].

If G is a l-group, the *lex kernel* of G is the join of all the minimal prime subgroups of G. Lavis [17] proved the following useful characterization of the lex kernel of an l-group G.

A *Lavis sequence* in an l-group G is a finite sequence of elements $g_1, g_2, ..., g_n$ such that $g_i \| g_{i+1}$ for all $i = 1, 2, ..., n-1$.

Theorem 2.5. For any l-group G, the lex kernel of G is the set $\{0\} \cup \{g \in G : g$ is an element of a Lavis sequence containing $0\}$.

We will not prove this theorem; the reader can find a proof in [17]. The reason for including the theorem is that we now have an immediately obvious proof of the Proposition 2.6 below. Our original proof was quite tedious and proceeded directly from the definition of the lex kernel; we would like to thank Paul Conrad for reminding us of Lavis' theorem and pointing out its immediate impact on this proposition.

Proposition 2.6. The lex kernel of an l-group is recognizable from the lattice and the identity.

Quite a few other structures of an l-group are easily proved to be recognizable from the lattice and the identity. Among these are the polars that are also direct summands of the group; these are called *cardinal summands.* From this fact we can easily derive two results due to Tsinakis [27] that show that whether or not an l-group is projectable or strongly projectable is decidable from the lattice. His theorems, however, are much stronger than this, giving actual lattice descriptions of projectable and strongly projectable l-groups.

Theorem 2.7. An element $s \in G$ is special if and only if s is not an element of the lex kernel of s''.

PROOF: For one direction of the proof, assume that s is special in G and let M be the value of s. Then $M \cap s''$ is the value of s in s'' [17]. Now if N is any minimal prime subgroup of s'', $N = P \cap s''$ for some minimal prime subgroup P of G such that $s'' \not\subset P$. Now $s \notin P$, else $s \in x'$ for some $x \notin P$ which implies that $x \in s'$ and so $s'' \subseteq x' \subseteq P$. Since $s \notin P, P \subseteq M$ and so $N \subseteq M \cap s''$. Thus the join of all minimal primes of s'' is in $M \cap s''$.

For the converse, if s is not in the lex kernel of s'', then s has only one value in s'' [17] and so has only one value in G.

Theorem 2.8. Special elements are recognizable from the lattice and the identity.

PROOF: s'' is of course recognizable from the lattice and the identity and within the lattice of s'', the lex kernel of s'' is recognizable from the lattice and the identity.

SECTION 3: The Quasitorsion Class of Special-valued l-groups

We start this section by reminding the reader of the fundamental theorem of special-valued l-groups as shown by Conrad in [15].

Theorem 3.1. An l-group G is special-valued if and only if every positive element is the join of a set of mutually disjoint positive special elements.

If G is special-valued and $0 < g \in G$, there exists a *unique* set of mutually disjoint positive special elements $\{g_\alpha : \alpha \in A\}$ in G such that $g = \bigvee g_\alpha$. Each such g_α is called a *special component* of g and $g = g_\alpha + \bigvee_{\beta \neq \alpha} g_\beta$ for all $\alpha \in A$. Equally important is that for all $\alpha \in A, g_\alpha \bigwedge \bigvee_{\beta \neq \alpha} g_\beta = 0$. These facts make finite lattice computations within a special-valued l-group very easy.

Proposition 3.2. Let G be special-valued and $0 < a, b \in G$. The special components of $a \vee b$ are the larger of all pairs of comparable special components of a and b and those special components of a disjoint from all special components of b and those special components of b disjoint from all special components of a. The special components of $a \wedge b$ are the lesser of all pairs of comparable special components of a and b.

The proof is quite clear and is omitted.

In [18], Conrad showed that the special-valued l-group form a quasitorsion class S, which, as said in the introduction, is a class of l-groups closed under taking convex l-subgroups, complete l-homomorphisms, and joins of convex l-subgroups. For S, the first condition is easily verified using Theorem 3.1; the second condition is well-known and easily proved. The third condition is harder to prove and in fact Conrad did not prove the condition directly. Instead, he showed that S is the intersection of two radical classes and thus is a radical class itself, which of course gives the third condition.

We will now give a direct proof of the third condition and this proof will also give us a description of the special-valued kernel of an l-group in terms of the lattice and the identity.

The only difficulty in the proof is that we are forced to make most of the computations in the lateral completion G^L of G. For the reader who may not be familiar with the lateral completion of an l-group, an l-group is laterally complete if every set of mutally disjoint positive elements has a least upper bound. Bernau [7] showed that every l-group G has a lateral completion G^L, unique up to l-isomorphisms, in which G is a dense l-subgroup. In the original paper on the lateral completion of an l-group, Conrad [16] showed that if $\Gamma(G)$ has a minimal plenary subset, so does $\Gamma(G^L)$. We will denote the minimal plenary subset, if it exists, of $\Gamma(G)$ by $\Delta(G)$ from now on in this paper. Conrad also showed that $\Delta(G^L)$ is order-isomorphic to $\Delta(G)$, where the order-isomorphism from $\Delta(G^L)$ onto $\Delta(G)$ is $H_\delta \to H_\delta \cap G = G_\delta$. Contained within this proof is the fact that this map has the property that for any $g \in G, g \in H^\delta \backslash H_\delta$ if and only if $g \in G^\delta \backslash G_\delta$.

Since we must deal with the lateral completion anyway, we will include the following results.

Lemma 3.3. If G if normal-valued and completely-distributive, each special element of G is also special in G^L.

PROOF: Since G is normal-valued and completely-distributive, $\Gamma(G)$ has a minimal plenary subset $\Delta(G)$, and so $\Gamma(G^L)$ has a minimal plenary subset $\Delta(G^L)$ which is order-isomorphic to $\Delta(G)$. Let g be a special element of G and M its value in G. Then $M \in \Delta(G)$ [17]. Now if H_{δ_1} and H_{δ_2} are distinct values of g in $\Delta(G^L)$, then $H_{\delta_1} \cap G$ and $H_{\delta_2} \cap G$ are distinct values of g in G. Thus the only value of g in G^L is that element of $\Delta(G^L)$ whose intersection with G is M.

Theorem 3.4. If G is special-valued, so is G^L.

PROOF: Let H be in the minimal plenary subset $\Delta(G^L)$ of $\Gamma(G^L)$. Then $G_\delta = H_\delta \cap G$ is special and we have seen that this forces H_δ to be special.

We have one more proposition to prove before we can show that S satisifes the third condition of quasitorsion classes.

Proposition 3.5. Let $0 < a = \bigvee_{\alpha \in A} a_\alpha$ and $b = \bigvee_{\beta \in B} b_\beta$ be joins of mutually disjoint positive special elements of an l-group G. Then $a + b$ is likewise a disjoint join of positive special elements.

PROOF: All that we need to do is to exhibit a set of mutually disjoint positive special elements of G whose join is $a+b$. We are thus free to compute in G^L, provided that at the end, all special elements do indeed belong to G. We will follow tradition and identity $\alpha \in A$ and $\beta \in B$ with the value of a_α or b_β, respectively. $A \cap B$ will indicate those values that lie in both A and B. For every $\alpha \in A$, let $B_\beta = \{\beta \in B : b_\beta << a_\alpha\}$ and let $A_1 = \{\alpha \in A : B_\alpha \neq 0\}$. Define A_β and B_1 similarly. Finally, define $A_2 = \{\alpha \in A : a_\alpha \wedge b_\beta = 0$ for all $\beta \in B$ and define B_2 analogously.

Then

$$a+b = \bigvee_{\alpha \in A} a_\alpha + \bigvee_{\beta \in B} b\beta$$

$$= \bigvee_{\alpha \in A \cap B} a_\alpha + \bigvee_{\alpha \in A_2} a_\alpha + \bigvee_{\beta \in B_1} \bigvee_{\alpha \in A} a_\alpha + \bigvee_{\alpha \in A_1} \bigvee_{\beta \in B_\alpha} b_\beta + \bigvee_{\beta \in B_2} b_\beta + \bigvee_{\beta \in A \cap B} b_\beta$$

$$= \bigvee_{\alpha \in A \cap B} (a_\alpha + b_\alpha) + \bigvee_{\alpha \in A_2} a_\alpha + \bigvee_{\beta \in B_2} b_\beta + \bigvee_{\beta \in B_1} [(\bigvee_{\alpha \in A_\beta} a_\alpha) + b_\beta] + \bigvee_{\alpha \in A_1} [a_\alpha + (\bigvee_{\beta \in B_\alpha} b_\beta)]$$

Now for any $\alpha \in A \cap B$, $a_\alpha + b_\alpha$ is special with value α. If $\beta \in B_1$, then $b_\beta \wedge a = b_\beta \wedge (\bigvee_{\alpha \in A} a_\alpha) = \bigvee_{\alpha \in A_\beta} a_\alpha$. Since each such a_α is in G_β and G_β is closed, $\bigvee_{\alpha \in A_\beta} a_\alpha \in G_\beta$ and so $\bigvee_{\alpha \in A_\beta} a_\alpha + b_\beta$ is special in G with value β. Similarly, for all $\alpha \in A_1$, $a_\alpha + (\bigvee_{\beta \in B_\alpha} b_\beta)$ is a special element of G.

Thus the set $\{a_\alpha + b_\alpha\}_{\alpha \in A \cap B} \cup \{a_\alpha\}_{\alpha \in A_2} \cup \{b_\beta\}_{\beta \in B_2} \cup \{a_\alpha + (\bigvee_{\beta \in B_\alpha} b_\beta)\}_{\alpha \in A_1} \cup \{(\bigvee_{\alpha \in A} a_\alpha) + b_\beta\}_{\beta \in B_1}$ is easily verified to consist of mutually disjoint positive special elements of G and $a+b$ is the join in G of this set.

The above proposition of course can be extended to any finite sum of joins of mutually disjoint positive special elements.

Theorem 3.6. Let $\{A_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{C}(G)$ be special-valued for all $\lambda \in \Lambda$. Then $\bigvee_{\lambda \in \Lambda} A_\lambda$ is special-valued.

PROOF: If $0 \leq g \in \bigvee_{\lambda \in \Lambda} A_\lambda$, $g = a_{\lambda_1} + a_{\lambda_2} + ... + a_{\lambda_n}$, where $a_{\lambda_i} \in \bigcup_{\lambda \in \Lambda} [A_\lambda +]$ [10]. But then g itself is a disjoint join of positive special elements. □

Now for any radical class $\mathcal{R}$ and any l-group G, there exists a unique largest convex l-subgroup $\mathcal{R}(G)$ which is called the *$\mathcal{R}$-kernel* of G [23]. (This $\mathcal{R}$-kernel is merely the join of all convex l-subgroups of G that are in $\mathcal{R}$.) Thus the torsion class $\mathcal{F}$ of finite-valued l-groups and the quasitorsion class $\mathcal{S}$ of special-valued l-groups have kernels within every l-group. Our next goal is to show that these are recognizable from the lattice and the identity.

Lemma 3.7. $0 \leq g \in G$ is in $\mathcal{S}(G)$ if and only if whenever $0 \leq h \leq g$, h is a disjoint join of positive special elements.

PROOF: If $g \in \mathcal{S}(G)$, then h is as well since $\mathcal{S}(G)$ is convex and so h is a disjoint join of positive special elements.

Conversely, let $0 \leq x \in G(g)$. Then $0 \leq x \leq ng$ for some positive integer n and so by the Riesz Decomposition Law [17], there exist $h_1, h_2, ..., h_n$ in G such that $0 \leq h_i \leq g$ and $x = h_1 + h_2 + ... + h_n$. But then each h_i is a disjoint join of positive special elements and thus x is as well. So $G(g)$ is special-valued and thus $g \in G(g) \subseteq \mathcal{S}(G)$. □

The above lemma is easily adapted to $\mathcal{F}(G)$:

Lemma 3.8. $0 \leq f \in G$ is in $\mathcal{F}(G)$ if and only if whenever $0 \leq h \leq g$, h is a disjoint join of finitely many positive special elements.

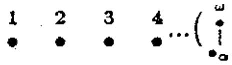
Theorem 3.9. $\mathcal{S}(G)$ and $\mathcal{F}(G)$ are recognizable from the lattice and the identity.

PROOF: Lemma 3.7 shows that $\mathcal{S}(G)^+$ is a convex submonoid regardless of the l-operations $+$ or $\oplus$ on $(G, \leq)$. Now for any positive special element g of G, both $-g$ and $\ominus g$ are negative special elements of G and so both $- \ominus g$ and $\ominus - g$ are positive special elements of G. Thus if $x \in \mathcal{S}(G)^+$, then both $- \ominus x$ and $\ominus - x$ are in $\mathcal{S}(G)^+$, since $x = \bigvee_{\lambda \in \Lambda} x_\lambda$ as a disjoint join of positive special elements and $- \ominus x = \bigvee_{\lambda \in \Lambda} - \ominus x_\lambda$. Thus the map $-\ominus$ permutes $\mathcal{S}(G)^+$ and so we are done.

A similar proof works for $\mathcal{F}(G)$. □

In Section 5, our main result is a new and easier proof that if G is normal-valued and δ is a normal plenary subset of $\Gamma(G)$, there exists a laterally complete special-valued l-group H whose minimal plenary subset of special values is order-isomorphic to Δ and such that there exists a value-preserving l-embedding of G into H. (The reader familiar with l-groups will recognize this as a generalization of the celebrated Conrad-Harvey-Holland theorem).

There exists, then a natural and intimate relation between $\mathcal{S}$ and $\mathcal{N}$, the l-variety of normal-valued l-groups, namely that $\mathcal{S} \subseteq \mathcal{N}$ and every l-group in $\mathcal{N}$ can be l-embedded in a value-preserving way into an element of $\mathcal{S}$. Since $\mathcal{N} \cdot \mathcal{N} = \mathcal{N}$, one might well then suspect that $\mathcal{S} \cdot \mathcal{S} = \mathcal{S}$. This is not the case and in fact, for any pair of ordinals $\alpha > \beta$, there exists an l-group $G \in \mathcal{S}^{\alpha} \backslash \mathcal{S}^{\beta}$. We will show examples of l-groups that are in $\mathcal{S}^2 \backslash \mathcal{S}$ and $\mathcal{S}^3 \backslash \mathcal{S}^2$, respectively, and then outline how one extends these examples for all succeeding cases. The first example is an example that has cropped up many times and many places, [10].

Example 3.1. Let $\Delta =$  and let G be the l-subgroup of $V(\Delta, Z)$ consisting of eventually constant integer sequences on Δ such that $\lim_{n\to\infty} g(h) = g(\omega)$ and where $g(\alpha)$ can be any integer. For any $\delta \in \Delta$, let $M_\delta = \{g \in G : g(\beta) = 0$ for all β in Δ such that $\beta \geq \delta$ (where the order is that of the root system Δ) $\}$. Each M_δ is a regular subgroup of G and if $\delta \neq \omega$, M_δ is special. But M_ω is not special, as if $g \notin M_\omega$, g must have support on some infinite subset of Δ and thus is not special. It is easy to see that $M_\omega = \mathcal{S}(G)$ and that $G/M_\omega \cong Z$. Thus $G \in \mathcal{S}^2 \backslash \mathcal{S}$.

To extend this example to an example of an l-group in $\mathcal{S}^3 \backslash \mathcal{S}^2$, we make each point in the upper tier of Δ above into a "foot" point of another copy of Δ like so:

*** Figure 1 here ***

On this root system, let G be the l-subgroup of $V(\Delta, Z)$ such that $g(\beta)$ is any integer, $\lim_{n\to\infty} g(\omega, n) = g(\omega, \omega)$, $\lim_{n\to\infty} g(\alpha, \omega)$, and $\lim_{n} g(n,k) = g(\omega, k)$. The special values of G then are $\{M_{m,n)} : m, n \in Z\}$, $\{M_{(m,\omega)} : m \in Z\}$, $\{M_{\alpha,n)} : n \in Z\}$, and M_β, these of course being defined as above. "Modding out" $\mathcal{S}(G)$ yields the previous example and so G is in $\mathcal{S}^3 \backslash \mathcal{S}^2$.

By making each point in the top tier of our new root system a "foot" point of another copy of Δ and choosing the necessary l-subgroup as indicated in the two above examples clearly yields an l-group in $\mathcal{S}^4 \backslash \mathcal{S}^3$. Building on this root system in the same way gives examples of l-groups in $\mathcal{S}^{n+1} \backslash \mathcal{S}^n$ for all positive integers n. Letting Δ_i denote the root system of $\mathcal{S}^i \backslash \mathcal{S}^{i-1}$, we have that Δ_i is an ideal of Δ_{i+1} for all positive integers i. By letting $\Omega = \bigcup_{i \in Z_+} \Delta_i$, we can choose an l-subgroup G_ω of $V(\Omega, Z)$ that is in $\mathcal{S}^\omega$ but not in $\mathcal{S}^n$ for any positive integer n. Taking a lexicographical product of G_ω with Z (lexing Z above G_ω, of course), gives an example of an l-group that is in

$S^{\omega+1}\backslash S^{\omega}$. We then can build as before and so get examples of l-groups G_α that lie in $S^{\alpha+1}\backslash S^{\alpha}$ for every ordinal α. One interesting fact that comes out of our construction for the limit ordinals is that not all special elements of G_α need be in $S(G)$.

We do not know whether or not the following theorems are true for transfinite powers of S. The theorems would be true for all powers of S if they are true for all limit ordinal powers.

Theorem 3.10. For every positive integer n, $S^n(G)$ is a closed l-ideal of G.

PROOF: We of course induct on n. The case $n = 1$ was shown by Conrad in [12], but we give our own proof now.

If $\{g_\alpha\}_{\alpha\in A} \subseteq S(G)^+$ and $g = \bigvee_{\alpha\in A} g_\alpha$ exists in G, then, since each g_α is the join of special elements, g is the join of special elements and so we can assume that each g_α is special. Let M_α be the value of g_α. Since $g \geq g_\alpha 0$, $g \notin M_\alpha$ implying that $M_\alpha \subseteq M$ a value of g. Since M_α is closed, so is M, and thus there exists a g_β such that g_β is not in M, either, and so $M = M_\beta$. So g has special values and hence special components.

Now suppose that $g \geq x \geq h_\beta$ for all special components h_β of g. Since g is in the order closure of $S(G)$, x is as well and so $g - x$ is also in the order closure of $S(G)$. Thus $g - x$,if not 0, has a special component t. Since $g > g - x \geq t$, g has a closed value containing the special value of t and this value must be special. Thus t is comparable to some special component h_β of g. But $0 < h_\beta \wedge t \leq h_\beta \wedge (g - x) \leq h_\beta \wedge (g - h_\beta) = 0$, and so we have a contradiction. Thus $g = x$ and so g is a disjoint join of special elements. Since this is true for all positive elements of $ocl(S(G))$, the order closure of $S(G)$ is special-valued and so contained in $S(G)$.

To show that this is true for $n + 1$ knowing that $S^n(G)$ is closed, we know that $S(G/S_n(G))$ is closed in $G/S^n(G)$ and from [11], this says that $S^{n+1}(G)$ is closed in G. □

Theorem 3.11. For any positive integer n and l-group $G \in S^n$, G is completely-distributive.

PROOF: Let $\Delta_1 = \{G_\delta \in \Gamma(G) : S^{n-1}(G) \subseteq G_\delta$ and $G_\delta/S^{n-1}(G)$ is in $\Delta(G/S^{n-1}(G))$. Let $\Delta_2 = \{G_\delta : G_\delta \cap S^{n-1}(G)$ is in $\Delta(S^{n-1}(G))\}$. Then each $M \in \Delta_1 \cup \Delta_2$ is closed and $\cap(\Delta_1 \cup \Delta_2) = (0)$.

Letting Δ be the dual ideal of $\Gamma(G)$ generated by $\Delta_1 \cup \Delta_2$ gives that Δ is a plenary subset of closed values and so G is completely-distributive. □

In the series of examples built in Example 3.1, none of the l-groups was archimedean. This was no accident.

Proposition 3.12. If G is an archimedean l-group and $G \in S^\alpha$ for some ordinal α, then G is special-valued.

PROOF: If $G \in S^\alpha$, then $S(G) \subseteq S^2(G) \subseteq ... \subseteq S^\alpha(G) = G$ and if $S^2(G) = S(G)$, the entire chain collapses to $S(G)$. So it suffices to show that if G is archimedean and $G \in S^2$, then G is special-valued.

So assume that $G \in S^2 \backslash S$. Then there is an $M \in \Delta(G)$ such that M is not a special value but $M/S(G)$ is a special-value in $G/S(G)$. Let $0 < g \in G$ such that $S(G)$ and g is special in $G/S(G)$. Then M is a value of g in G and so g is not special. Thus g has another value N in $\Delta(G)$ [17] and $S(G) \not\subseteq N$, else $S(G) + g$ has two distinct values in $G/S(G)$.

Since $S(G)$ such that N, there exists $0 < h \in S(G)/N$ and so, as seen before, N is contained in a special value of h. But then N itself must be *the* value of a special element of $S(G)$. Thus g has special components which are in $S(G)$. Since $S(G)$ is a closed l-ideal of G, g can not be the join of its special components that are in $S(G)$.

Let $0 < x \in G$ such that $g > x \geq h_\beta$ for every special component h_β of g that is an element of $S(G)$. If $g - x$ has a special value, $g - x$ has a special component $t > 0$; note that $g > g - x \geq t > 0$. But for any special component h_β of g in $S(G)$, $0 \leq t \wedge h_\beta \leq (g - x) \wedge h_\beta \leq (g - h_\beta) \wedge h_\beta = 0$, and so t must be an element of M. But then $nt \in M$ for any integer n and so $nt < g$ for all integers n. Thus G is not archimedean.

If $g - x$ has no special values, let $0 < y \in G$ be such that $S(G) + y$ is a special component of $S(G) + g - x$ in $G/S(G)$. Then y is not special in G and so has special components that are in $S(G)$. Let t be one such special component of y. Then $g - x \geq y > t > 0$ and since $g - x$ has no special values, t must be an element of a closed value of $g - x$ and again, then, $g - x > nt$ for every integer n. □

The final part of this section will discuss the relation between the closed convex l-subgroups of a special-valued l-group G and the minimal plenary subset of $\Gamma(G)$. Conrad [14] showed that an l-group G is finite-valued if and only if $\Delta(G) = \Gamma(G)$ and if and only if the lattice of convex l-subgroups is freely generated by the regular subgroups (we define 'freely generated' below), thus proving there exists a one-to-one order-reversing correspondence between the dual ideals of $\Gamma(G)$ and the convex l-subgroups of G. In a later paper [18], he showed that if the closed convex l-subgroups of G are freely generated by the closed regular subgroups, G must be special-valued. Anderson and Conrad [1] then studied the closed convex l-subgroups of $V(\Delta, \Re)$, showing that there exists a one-to-one order-preserving correspondence between the ideals of Δ and the closed convex l-subgroups of $V(\Delta, \Re)$. Moreover, if Λ is an ideal of Δ and K is the associated closed convex l-subgroup, then $K \cong V(\Lambda, \Re)$ and $V(\Delta, \Re)/K$ is l-isomorphic to $V(\Delta \backslash \Lambda, \Re)$.

Our next theorem is that an l-group is special-valued if and only if the lattice of closed convex l-subgroups is freely generated by the closed regular subgroups. This generalizes Conrad's theorem in [14] from finite-valued to special-valued l-groups and incorporates his result from [18]. Our theorem then implies Anderson's and Conrad's theorem about closed l-ideals of $V(\Delta, \Re)$.

An element M of a lattice L is *meet-irreducible* if $M = \bigwedge_{\alpha \in A} \mathcal{N}_\alpha \Rightarrow \mathcal{M} = \mathcal{N}_\alpha$ for some $\alpha \in A$. The meet-irreducible elements of $\mathcal{C}(G)$ are the regular subgroups [17] and the meet-irreducible elements of $\mathcal{K}(G)$ are the closed regular subgroups [11]. For any lattice L, the meet-irreducible elements form a root system under the order inherited from L. The lattice L is *generated* by the root system of meet-irreducible elements if every element of L is the meet of a dual ideal of the root system and L is *freely generated* by the meet-irreducible elements if for each element l of L, there exists a *unique* dual ideal of the root system whose meet is l.

Theorem 3.13. An l-group G is special-valued if and only if the lattice of closed convex l-subgroups is freely generated by the closed regular subgroups.

PROOF: Let G be special-valued and M be a closed regular subgroup of G. Let $0 < g$ be an element of G such that M is the value of g; then, as seen before, M is the value of a special component of g and so the closed regular subgroups of G are precisely the special values.

Now let $K \in \mathcal{K}(G)$. Then $K = \cap\{G_\delta \in \Delta(G) : K \subseteq G_\delta\}$ and this set is clearly a dual ideal of $\Delta(G)$. So $\Delta(G)$ generates $\mathcal{K}(G)$. If $\Lambda \subseteq \Delta(G)$ is a dual ideal and $K = \cap\{G_\delta : G_\delta \in \Lambda\}$, clearly $K \in G_\delta$ for all $G_\delta \in \Lambda$ and so $\Lambda \subseteq \{G_\delta \in \Delta(G) : K \subseteq G_\delta\}$. Suppose there is a $G_\delta \in \Delta(G)$ such that $K \subseteq G_\delta \notin \Lambda$. Let $0 < g_\delta$ be special with value G_δ. Then for all $G_\lambda \in \Lambda$, $g_\delta \in G_\lambda \Rightarrow g_\delta \in \cap\{G_\lambda : \lambda \in \Lambda\} = K \subseteq G_\delta$ a contradiction.

Conversely, suppose $\mathcal{K}(G)$ is freely generated by the closed regular subgroups of G. (This part is due to Conrad and appeared in [18].) Let G_δ be a closed regular subgroup. Then if $G_\delta \subseteq G_\alpha$, G_α is also closed [13] and so the closed regular shape form a dual ideal of $\Gamma(G)$. Since $(0) \in \mathcal{K}(G)$, (0) is then the intersection of all closed regular subgroups and so these form a plenary subset of $\Gamma(G)$.

Now for any closed regular subgroup G_δ of G, the sets of closed regular subgroups $\{G_\alpha : G_\alpha \not\subseteq G_\delta\}$ and $\{G_\beta : G_\beta \not\subset G_\delta\}$ are distinct dual ideals of the closed regular subgroups and thus have distinct intersections. This implies that there exists $0 < g \in \cap\{G_\alpha : G_\alpha \not\subseteq G_\delta\} \setminus \cap\{G_\beta : G_\beta \not\subset G_\delta\}$ and G_δ is clearly the only value of g. So G_δ is special. □

This theorem of course states that if G is special-valued, then there exists a one-to-one order-reversing correspondence between the closed convex l-subgroups of G and the dual ideals of $\Delta(G)$. The correspondence is as follows: if $K \in \mathcal{K}(G)$, then the corresponding dual ideal Λ consists of those closed regular subgroups that are not values of elements of K and if Λ is a dual ideal of $\Delta(G)$, the associated closed convex l-subgroup are those elements of G whose special values are not in Λ.

Recasting the above in terms of ideals of $\Delta(G)$ gives us the following generalization of the Anderson and Conrad theorem.

Corollary 3.14. An l-group is special-valued if and only if there exists a one-to-one order-preserving correspondence between the closed convex l-subgroups of G and the ideals of the root system of closed regular subgroups, where the correspondence takes $K \in \mathcal{K}(G)$ to all special-values of elements of K.

Corollary 3.15. Let G be special-valued and K be a closed l-ideal of G. Then $\Delta(G/K)$ is order isomorphic to the dual ideal of $\Delta(G)$ associated with K.

To get our final corollary to Theorem 3.13, we need the definition of an *a*-extension* of an l-group.

Let H be a l-group and G an l-subgroup of H. H is an *a*-extension* of G if whenever K_1 and K_2 are two distinct closed convex l-subgroups of H, $K_1 \cap G \neq K_2 \cap G$ [11]. It is known that if $K \in \mathcal{K}(H)$, then $K \cap G \in \mathcal{K}(G)$ [11] so what an a*-extension insists upon is that the correspondence be one-to-one.

Corollary 3.16. If G is special-valued, then G^L is an a*-extension of G.

PROOF: We know that G^L is special-valued and that $\Delta(G^2)$ is order-isomorphic to $\Delta(G)$, when $H_\delta \in \Delta(G^2)$ is mapped to $H_\delta \cap G$. Thus if K_1 and K_2 are distinct closed convex l-subgroups of G^2 with corresponding dual ideals Λ_1 and Λ_2 of $\Delta(G^2)$, $\Lambda_1 \neq \Lambda_2$ and so $K_1 \cap G \neq K_2 \cap G$ since these arise from the same dual ideals of $\Delta(G)$. □

SECTION 4: The Main Theorem

The main result of this section is that a laterally complete, completely distributive, normal-valued l-group must be special-valued. In the next section we use this result to gain a generalization of the famous Conrad-Harvey-Holland embedding theorem for abelian l-groups, namely, that every normal-valued l-group can be l-embedded into a special-valued l-group. This generalization was first proved in [5], but our proof is simpler and more direct.

The next several lemmas set up the proof. Throughout these lemmas, let Δ be a plenary subset of $\Gamma(G)$ and let $M(g) = \{\delta \in \Delta : g \in G^\delta \backslash G_\delta\}$ be the set of values of g in Δ.

Lemma 4.1. If $0 < x$, $y \in G$ are such that $M(y) \subseteq M(x)$ and for all $\delta \in M(y)$, $y + G_\delta > x + G_\delta$, then $[x - (x \wedge y)] \wedge [x \wedge y] = 0$.

PROOF: For any $0 < g$, $h \in G$ it is true that $M(g) \cap M(h) \subseteq M(g \wedge h)$ and $M(g) \cap M(h) = 0$ if and only if $g \wedge h = 0$. Thus, under the hypothesis above, $M(y) \subseteq M(x \wedge y)$. Let $\delta \in M(x \wedge y)$. Since Δ is plenary and $y \notin G_\delta$, there is $\alpha \in M(y)$ with $\alpha \geq \delta$. Since $M(x \wedge y)$ is trivially ordered, $\alpha = \delta$ and so $\delta \in M(y)$. By hypothesis then,

$$\begin{aligned} G_\delta < y - x + G_\delta &= (y + G_\delta) - (x + G_\delta) \\ &\leq (y + G_\delta) - (x \wedge y + G_\delta) \\ &= y - (x \wedge y) + G_\delta \end{aligned}$$

Since $y \in G^\delta$, $y - (x \wedge y) \in G^\delta \backslash G_\delta$ and so $M(x \wedge y) \subseteq M(y - x \wedge y)$. But now,$(x - x \wedge y) \wedge (y - x \wedge y) = 0$ implies $\emptyset = M(x - x \wedge y) \cap M(y - x \wedge y) \supseteq M(x - x \wedge y) \cap M(x \wedge y)$ and so $[x - x \wedge y] \wedge [x \wedge y] = 0$.

Lemma 4.2. Let $0 < x$, $y \in G$. Define $w_n = [(n+2)y - x]^+ \wedge [x - ny]^+$. If $w_n \neq 0$, then every value of w_n is less than or equal to some mutual value of x and y.

PROOF: Let α be a value of w_n. Since $x \geq [x - my]^+ \geq w_n > 0$, we must have $x \notin G_\alpha$. Thus x has a value $\delta_x \geq \alpha$. Similarily $(n+2)y \geq [(n+2)y - x]^+ \geq w_n > 0$ and so y has a value $\delta_y \geq \alpha$.

Since the regular subgroups form a root system we know that δ_x and δ_y are comparable. To show that they are equal first suppose that $\alpha \leq \delta_x < \delta_y$. As above, $[(n+2)y - x]^+$, $[x - ny]^+ \notin G_\alpha$ and thus, since G_α is regular, $[(n+2)y - x]^-$, $[x - ny]^- \in G_\alpha \subseteq G_{\delta_x} \subset G_{\delta_y}$. If $x \in G_{\delta_y}$, then $[x - ny]^+ \in G_{\delta_y}$ and so $x - ny \in G_{\delta_y}$ which implies that $y \in G_{\delta_y}$, a contradiction. Thus $x \notin G_{\delta_y}$ and so by maximality of G_{δ_x}, $\delta_y = \delta_x$.

A similar argument applies to the assumption that $\alpha \leq \delta_y < \delta_x$.

The proof of the next lemma uses a technique given in [8] and [2].

Lemma 4.3. Let G be laterally complete, completely distributive, and normal-valued. For any elements $0 < x, y \in G$, there exists a component z of x such that $M(z) = M(x) \cap M(y)$.

PROOF: For each nonnegative integer n define $w_n = [(n+2)y - x]^+ \wedge [x - ny]^+$. Notice that $w_{2n} \wedge w_{2m} = 0 = w_{2n+1} \wedge w_{2m+1}$ whenever $n \neq m$.

To see this suppose, without loss of generality, that $m > n$. Then $m \geq n+1$ and so $2m \geq 2n+2$. Thus

$$\begin{aligned} w_{2m} \wedge w_{2n} &= [(2m+2)y - x]^+ \wedge [x - 2my]^+ \wedge [(2n+2)y - x]^+ \wedge [x - 2ny]^+ \\ &\leq [x - 2my]^+ \wedge [(2n+2)y - x]^+ \\ &\leq [x - (2n+2)y]^+ \wedge [(2n+2)y - x]^+ \\ &= 0 \end{aligned}$$

Let $u = \bigvee_{n=0}^{\infty} (2n+1)w_{2n}$, $v = \bigvee_{n=0}^{\infty} (2n+2)w_{2n+1}$. Both of these exist since G is laterally complete. Let $\delta \in M(x) \cap M(y)$.

Case 1: $G_\delta < x + G_\delta \leq y + G_\delta$. Then

$$\begin{aligned} 2x + G_\delta \leq 2y + G_\delta &\Rightarrow x + G_\delta \leq 2y - x + G_\delta \\ w_0 + G_\delta = [2y - x]^+ \wedge [x]^+ + G_\delta &= [(2y - x) \wedge x] \bigvee 0 + G_\delta \\ &= [(2y - x + G_\delta) \wedge (x + G_\delta)] \bigvee G_\delta = x + G_\delta \end{aligned}$$

and $w_k + G_\delta = G_\delta$ for all $k \geq 1$. Thus $(u + v) + G_\delta = x + G_\delta$.

Case 2: $G_\delta < y + G_\delta \leq x + G_\delta$. Since G^δ / G_δ is isomorphic to a subgroups of the real numbers, there is an integer $k > 0$ such that $G_\delta < ky + G_\delta \leq x + G_\delta$ and $G_\delta < x + G_\delta \leq (k+1)y + G_\delta$. Thus $2ky + G_\delta \leq 2x + G_\delta \Rightarrow (k+1)y - x + G_\delta \leq x - (k-1)y + G_\delta$ and $2x + G_\delta \leq 2(k+1)y + G_\delta \Rightarrow x - ky + G_\delta \leq (k+2)y - x + G_\delta$. Thus we have

$$\begin{aligned} w_{k-1} + G_\delta &= ([(k+1)y - x]^+ \wedge [x - (k-1)y]^+) + G_\delta \\ &= ([(k+1)y - x] \wedge [x - (k-1)y]) + G_\delta \\ &= [(k+1)y - x] + G_\delta \end{aligned}$$

Also

$$\begin{aligned} w_k + G_\delta &= ([(k+2)y - x]^+ \wedge [x - ky]^+) + G_\delta \\ &= x - ky + G_\delta \end{aligned}$$

If $m \leq k-2$, $(m+2)y - x + G_\delta \leq ky - x + G_\delta \leq G_\delta$ and if $m > k$, $x - my + G_\delta \leq x - (k+1)y + G_\delta < G_\delta$. That is, $w_m \in G_\delta$ if $m \neq k$ and $m \neq k-1$. Thus

$$\begin{aligned} (u+v) + G_\delta &= kw_{k-1} + G_\delta + (k+1)w_k + G_\delta \\ &= k(k+1)y - kx + (k+1)x - k(k+1)y + G_\delta \\ &= x + G_\delta \end{aligned}$$

We have thus shown that for each $\delta \in M(x) \cap M(y)$, $u + v + G_\delta = x + G_\delta$ and so $M(x) \cap M(y) \subseteq M(u+v)$. By *Lemma 4.2.* we have that for all $\gamma \not\leq$ any $\alpha \in M(x) \cap M(y)$, $w_k \in G_\gamma$ for all k. Since G_γ is closed, $u + v \in G_\gamma$ for all such γ as well. Thus $\gamma \in M(u+v) \Rightarrow \gamma \leq$ some $\alpha \in M(x) \cap M(y) \subseteq M(u+v)$ and since $M(u+v)$ is trivially ordered, we conclude that $M(u+v) = M(x) \cap M(y)$. In particular $M(2(u+v)) = M(u+v) \subseteq M(x)$ and for all $\delta \in M(2(u+v))$, $2(u+v) + G_\delta \geq x + G_\delta$. Let $z = [x \wedge 2(u+v)]$ and notice that $M(z) = M(u+v)$. By *Lemma 4.1.* $z \wedge (x - z) = 0$ and the proof is complete.

Lemma 4.4. Let $\delta_1 \| \delta_2$ be elements of $\Gamma(G)$. There is an element $0 < g \in G$ with $\delta_1 \in M(g)$ and $\delta_2 \| M(g)$.

PROOF: Let $0 < g_1 \in G_{\delta_1} \backslash G_{\delta_2}$ and $0 < g_2 \in G_{\delta_2} \backslash G_{\delta_1}$. Since $\delta_1 \| \delta_2$, $G_{\delta_1} \cap G_{\delta_2}$ is not prime. Thus there exists $x, y \in G$ such that $x \wedge y = 0$, but neither x nor y is in $G_{\delta_1} \cap G_{\delta_2}$. Without loss of generality $x \in G_{\delta_1} \backslash G_{\delta_2}$ and $y \in G_{\delta_2} \backslash G_{\delta_1}$. Let $0 < t \in G^{\delta_2} \backslash G_{\delta_2}$ and let $0 < s \in G^{\delta_1} \backslash G_{\delta_1}$. Then

$t\wedge x \notin G_{\delta_2}$ and $s\wedge y \notin G_{\delta_1}$, because both G_{δ_1} and G_{δ_2} are prime. Also, $0 \leq (t\wedge x)\wedge(s\wedge y) \leq x\wedge y = 0$. It is now easy to see that the element $g = s \wedge y$ satisfies the condition of the lemma.

We can now prove our main result.

Theorem 4.5. The lateral completion of a completely distributive, normal-valued l-group is special-valued.

PROOF: Let G be completely distributive, normal-valued with minimal plenary subset Δ. Then G^L is also completely distributive, normal-valued and has Δ as its minimal plenary subset. Let $\delta \in \Delta$, $0 < x \in G^\delta \backslash G_\delta$. If x is not special, then there is $\delta_1 \neq \delta$ in $M(x)$. By Lemma 4.4, there is an element $y_1 \in G$ such that $\delta \| M(y_1)$ and $\delta_1 \in M(y_1)$. Let $A_1 = M(x) \cap M(y_1)$. By Lemma 4.3, there is a component, z_1, of x with $M(z_1) = A_1$. Also, notice that $\delta \in M(x - z_1) \subsetneq M(x)$ and $\delta_1 \in M(x)\backslash M(x - z_1)$. Now let β be any ordinal and suppose we have defined δ_α and z_α as above for all $\alpha < \beta$ such that (i) $\bigvee_{\alpha<\beta} z_\alpha$ exists and is a component of x, (ii) $\delta \in M(x - \bigvee_{\alpha<\beta} z_\alpha) \subsetneq M(x)$, and (iii) δ_α are for all $\alpha < \beta$ distinct members of $M(x)\backslash M(x - \bigvee_{\alpha<\beta} z_\alpha)$. If $x - \bigvee_{\alpha<\beta} z_\alpha$ is not special then it has a value $\delta_\beta \neq \delta$. Again by Lemmas 4.3 and 4.4 there is a component z_β of $x - \bigvee_{\alpha<\beta} z_\alpha$ with $\delta \in M(x - \bigvee_{\alpha<\beta} z_\alpha - z_\beta) \subsetneq M(x)$ and $\delta_\beta \in M(x)\backslash M(x - \bigvee_{\alpha<\beta} z_\alpha - z_\beta)$. Notice also that z_β is disjoint from $\bigvee_{\alpha<\beta} z_\alpha$ and so $x - \bigvee_{\alpha<\beta} z_\alpha - z_\beta = x - \bigvee_{\alpha\leq\beta} z_\alpha$. Clearly δ_β is distinct from δ_α for all $\alpha < \beta$ and $\bigvee_{\alpha\leq\beta} z_\alpha$ is a component of x. Finally, since for each $\alpha < \beta$, $\delta_\alpha \notin M(x - \bigvee_{\alpha<\beta} z_\alpha)$ and $x - \bigvee_{\alpha\leq\beta} z_\alpha$ is a component of $x - \bigvee_{\alpha<\beta} z_\alpha$, we have that for all $\alpha \leq \beta$ the δ_α are distinct elements of $M(x)\backslash M(x - \bigvee_{\alpha\leq\beta} z_\alpha)$.

Now since $M(x)$ has a fixed cardinality, we must have that $x - \bigvee_{\alpha<\beta} z_\alpha$ is special at δ for some ordinal β. That is, $G^L \in \mathcal{S}$.

Corollary 4.6. If G is completely distributive and normal valued, then G is special-valued if and only if G^L is an a*-extension of G.

PROOF: ($\rightarrow$) This is Corollary 3.16 ($\leftarrow$) By Theorem 4.5, G^L is special-valued and so by Theorem 3.13 $K_1(G^L)$ freely generates $K(G^L)$. Since $K \rightarrow K \cap G$ is a one-to-one correspondence between $K(G^L)$ and $K(G)$, $K_1(G)$ also freely generates $K(G)$. Thus G is special valued.

Corollary 4.7. For a normal-valued l-group G, the following and equivalent:

a. G is completely-distributive

b. G^L is special-valued

c. (Ball and Davis) G can be embedded as a dense l-subgroup of a special-valued l-group.

SECTION 5: A Generalization of the Conrad-Harvey-Holland Theorem

Probably the most fundamental theorem of abelian l-groups is the Conrad-Harvey-Holland Theorem [19] which says that for any abelian group G and any plenary subset Δ of $\Gamma(G)$, there exists a l-embedding τ of G into $V(\Delta, \Re)$ such that for any $\delta \in \Delta$, $g \in G^{\delta} \backslash G_{\delta}$ if and only if $\tau(g) \in V^{\delta} \backslash V_{\delta}$. In this case, τ is called a *v-isomorphism.*

In [5], Ball, Conrad, and Darnel gave this generalization of the theorem:

Theorem 5.1. Let G be a normal-valued l-group and Δ be a normal plenary subset of $\Gamma(G)$. There exists a laterally complete special-valued H such that $\Delta(H) \cong \Delta$ and there exists a v-isomorphism of G into H.

Their proof used several constructions previously developed by Ball ([3] and [4]) and Ball and Davis [6] with which many persons in the field are not familiar. They also did not show the hypothesis that Δ be a *normal* plenary subset was necessary. The proof we give in this section is much simpler and more direct than the proof of [5], depending only on our main theorem of Section Four and the following construction due to Bigard, Conrad, and Wolfenstein [9] that embeds any l-group into an l-group with a basis.

Let G be a normal-valued l-group and Δ a normal plenary subset of $\Gamma(G)$. Let $\mathcal{A}$ be the set of all roots C of Δ and $B = \sum_{C \in \mathcal{A}} Z_C$. $\forall g \in G$, let $\sigma(g)$ be the l-automorphism of B defined by, if $C \in \mathcal{A}$ and $b \in B$, the C-component of $\sigma(g)(b)$, $\sigma(g)(b)_C$, equals the gCg^{-1} component of b. ($\sigma(g)$ is thus a "shift" l-automorphism of B.) On the set $G \times B$, define $(g_1, b_1) + (g_2, b_2) = (g_1 + g_2, \sigma(g_2)(b_1) + b_2)$, and define $(g, b) \geq (0,0)$ if $g \geq 0$ and if the projection of b onto $\{C \in \mathcal{A} : g \in \cap C\}$, called $\hat{g}(b)$, is positive. $(G \times B, \leq, +)$ is then an l-group. In [5], the values of (g, b) are shown to be of the form $C \times B$, where C is a value of g in G, and $G_D \times D$, where D is a value of $\hat{g}(b)$ in B and $G_D = \{h \in G : h \in \cap C$ for all $C \in \mathcal{A}$ such that $b_C = 0$ for all $b \in D\}$. Moreover, the maps $\alpha : G \to G \times B : g \to (g, 0)$, $\beta : B \to G \times B : b \to (0, b)$ are l-embeddings and for any $\delta \in \Delta$, $g \in G^{\delta} \backslash G_{\delta}$ if and only if $(g, 0) \in (G^{\delta} \times B) \backslash (G_{\delta} \times B)$, and for any $C \in \mathcal{A}$, b is special in B with value $D = \sum_{C' \neq C} Z_{C'}$ if and only if $(0, b)$ is basic in $G \times B$ with value $G_D \times D$. $G \times B$ then has a basis and so is completely-distributive. Since $G \times B \in \mathcal{N} \cdot \mathcal{N} = \mathcal{N}$[22], $\Gamma(G \times B)$ has a minimal plenary subset $\Delta(G \times B)$ which is simply a copy of Δ with a new regular subgroup added under each root of Δ.

The proof of Theorem 5.1. is now easy. Let $H = (G \times B)^L$. Then H is special-valued and $\Delta(H) \cong \Delta(G \times B)$. Let K be the closed convex l-subgroup of H associated with the lowest tier of elements of $\Delta(H)$ that were added to Δ in building $\Delta(G \times B)$. Since $(0) \times B$ is an l-ideal of $G \times B$, K is an l-ideal of H. By Proposition 3.15, $\Delta(H/K) \cong \Delta$. Letting σ denote the embedding of $G \times B$ into H and η the natural homomorphism of H onto H/K, for any $\delta \in \Delta$, $g \in G^\delta/G_\delta$ if and only if $\eta \circ \sigma \circ \alpha(g) \in \frac{H^\delta/K}{H_\delta/K}$.

Corollary 5.2. (J. Reed) An l-group G is normal-valued if and only if G if l-isomorphic to an l-subgroup of a special-valued l-group.

The following example shows that the hypothesis of Theorem 5.1 that Δ be a *normal* plenary subset of $\Gamma(G)$ can not be discarded.

Let $G = \Re \overrightarrow{\times} C(\Re)$, where $(y, f(x)) \geq (0, 0)$ if $y > 0$ or if $y = 0$ and $f(x) > 0$ for all $x \in \Re$, and where $(y_1, f_1(x)) + (y_2, f_2(x)) = (y_1 + y_2, h(x))$, where $h(x) = f_1(x + y_2) + f_2(x)$. G is then an l-group [17]. Now since Q is a dense subset of $\Re$, every $f(x) \in C(\Re)$ is determined by its action on Q. This implies that the set of maximal convex l-subgroups [20] $M_q = \{f \in C(\Re) : f(q) = 0\}$, and $q \in Q$, is a plenary subset of $\Gamma(C(\Re))$, and consequently the set

$$\Delta = \begin{array}{c} C(\Re) \\ / \; / \; | \; \backslash \; \backslash \\ \{M_q : q \in Q\} \end{array}$$

is a plenary subset of $\Gamma(G)$.

Now if Theorem 5.1 were to hold for G, there would exist an l-embedding τ of G into $H = \Re \overrightarrow{\times} \prod_{q \in Q} \Re_q$, with some strange group operation on H_q such that if q has value M_q, $\tau(q)$ has value $H_q = \{h \in \prod_{q \in Q} \Re_q : h(q) = 0\}$.

Let $0 < h \in \Re_0$ in H; *the* value of h, then, is $H_0 = \{h \in \prod_{q \in Q} \Re_q : h(0) = 0\}$, and *the* value of any conjugate of h must be of the form H_q for some $q \in Q$. Thus any set of pairwise disjoint conjugates of h must be countable.

Let y, z be distinct real numbers and $0 < \epsilon < |y - z|$. Define $h_\epsilon \in H$ by

$$h_\epsilon(q) = \begin{cases} 0, & q < -\epsilon/2 \\ 2h/\epsilon q + h, & -\epsilon \leq q < 0 \\ -2h/\epsilon q + h, & 0 \leq q < \epsilon/2 \\ 0, & \epsilon/2 \leq q \end{cases}$$

for $q \in Q$. Then h is a special component of h_ϵ. Note that the support of h_ϵ is the set of rational points in the interval $(-\epsilon/2, \epsilon/2)$.

But now in H, $(y,0) + (0, h_\epsilon) - (y,0)$ is disjoint from $(z,0) + (0, h_\epsilon) - (z,0)$, implying of course that $(y,0) + h - (y,0)$ is disjoint from $(z,0) + h - (z,0)$. Since this is true for any two distinct real numbers y and z, the set $\{(x,0) + h - (x,0) : x \in \Re\}$ is a pairwise disjoint set of conjugates of h, contradicting the fact that such a set must be countable.

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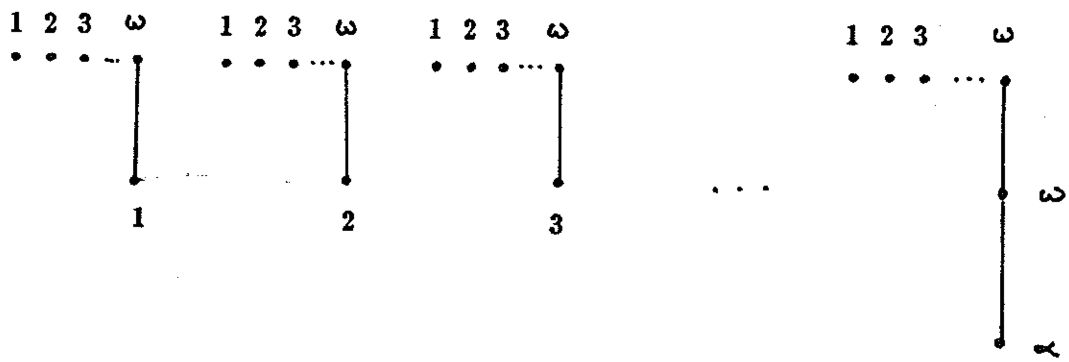

figure 1