TECHNICAL REPORT CS81008-R

Overhang of a Heavy Elastic Sheet

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Abstract

A flexible elastic sheet overhangs from a corner. The deflection due to its own weight depends on a parameter K which represents the relative importance of overhang length to the bending length $(EI/\rho)^{1/3}$.

Zusammenjassung

Ein biegsames Blech überhängt eine Ecke. Die vom Eigengewicht verursachte Abbiegung hängt von einem Parameter K ab der die relative Wichtigkeit der Überhängungslänge zur Biegesteife darstellt.

1. Introduction and Formulation

The overhang of a semi-infinite elastic sheet over a corner is important in structural engineering and in the textile and paper industries. Figure 1 shows such an elastic sheet freely resting on a semi-infinite rigid foundation at $x' \geq 0$. Due to the weight of the overhang, the sheet is raised and separated from the foundation in the segment from the corner 0 to the point of contact at $x' = x'_c$. We assume the corner offers little frictional resistance. The sheet is kept in equilibrium by the horizontal force H' at x'_c. This horizontal force may be due to frictional resistance of the semi-infinite segment of contact $x' \geq x'_c$.

Let s' be the arc length from 0 and ℓ be the length of the overhang. The sheet can be divided into three segments: the overhang from $s' = -\ell$ to $s' = 0$, the raised segment from $s' = 0$ to $s' = k'$, and a contact segment $s' \geq k'$ (where $x' \geq x'_c$). Since the force must be normal to the sheet at the point 0, the vertical force there (F') is related to H' by

$$\tan \alpha = \frac{H'}{F'} \tag{1}$$

where α is the angle of inclination at 0. If ρ is the weight per unit length, the vertical force G' at the point of contact $s' = k'$ is then

$$G' = (\ell + k')\rho - F' \tag{2}$$

A local balance of momentum (Figure 1) gives, for the overhang segment,

$$m + dm = m - \rho(\ell + s') \cos\theta \, ds' \tag{3}$$

Here m is the local moment, and θ is the local angle of inclination. If the sheet is thin enough, the local moment is proportional to the local curvature:

$$m = EI \frac{d\theta}{ds'}, \tag{4}$$

where EI is the flexural rigidity. We normalize all lengths by ℓ and drop primes. Eqs. (3, 4) become

$$\frac{d^2\theta}{ds^2} = -K(1 + s) \cos\theta \tag{5}$$

where $K = \rho\ell^3/EI$ represents the relative importance of density and length to flexural rigidity. The boundary conditions are

$$\theta(0) = \alpha , \quad \frac{d\theta}{ds}(0) = \lambda \tag{6}$$

$$\frac{d\theta}{ds}(-1) = 0 \tag{7}$$

Similarly, the equation for the raised segment is

$$\frac{d^2\theta}{ds^2} = [F - K(1 + s)] \cos\theta + F \tan\alpha \sin\theta \tag{8}$$

Here all forces have been normalized by EI/ℓ^2. The shape of the sheet is given by

$$\frac{dx}{ds} = \cos\theta \ , \quad \frac{dy}{ds} = \sin\theta \tag{9}$$

with the boundary conditions

$$x(0) = y(0) = 0 \ , \quad \theta(0) = \alpha \ , \quad \frac{d\theta}{ds}(0) = \lambda \tag{10}$$

$$y(k) = \theta(k) = \frac{d\theta}{ds}(k) = 0 \tag{11}$$

Given K, Eqs. (5 - 11) are to be solved concurrently for the unknowns α, λ, F, k.

<u>The solution for small K</u>

Small K signifies low density, short length or large rigidity. We expect θ, α, λ, F to be small also. We expand

$$F = KF_o + 0(K^3) \ , \quad \theta = K\theta_o + 0(K^3) \ , \ y = Ky_o + 0(K^3) \tag{12}$$

$$\alpha = K\alpha_o + 0(K^3) \ , \ \lambda = K\lambda_o + 0(K^3) \ , \ k = k_o + 0(K^2) \tag{13}$$

Then Eqs. (5, 6) can be approximated by

$$\frac{d^2\theta_o}{ds^2} = -(1 + s) , \frac{d\theta_o}{ds} (-1) = 0 , \theta_o(0) = \alpha_o, \frac{d\theta_o}{ds} (0) = \lambda_o \qquad (14)$$

The solution for the cantilever segment is

$$\theta_o = \alpha_o - \frac{s}{2} - \frac{s^2}{2} - \frac{s^3}{6} , \lambda_o = -\frac{1}{2} \qquad (15)$$

Similarly from Eqs. (7 - 11) we find

$$\frac{d^2\theta_o}{ds^2} = -(1 + s) + F_o , \frac{dy_o}{ds} = \theta_o , \qquad (16)$$

$$\theta_o(0) = \alpha_o , \frac{d\theta_o}{ds} (0) = \lambda_o , y_o(0) = 0 \qquad (17)$$

$$\theta_o(k_o) = 0 , \frac{d\theta_o}{ds} (k_o) = 0 , y_o (k_o) = 0 \qquad (18)$$

The solution for the raised segment is

$$\theta_o = \alpha_o - \frac{s}{2} + (F_o - 1) \frac{s^2}{2} - \frac{s^3}{6} \qquad (19)$$

$$y_o = \alpha_o s - \frac{s^2}{4} + (F_o - 1) \frac{s^3}{6} - \frac{s^4}{24} \qquad (20)$$

From the boundary conditions the unknowns are found to be

$$k_o = \sqrt{2} \quad , \quad \alpha_o = \frac{1}{6\sqrt{2}} \quad , \quad F_o = \frac{3}{2\sqrt{2}} + 1 \tag{21}$$

Therefore

$$\alpha = \frac{1}{6\sqrt{2}} K + 0(K^3) \tag{22}$$

$$\lambda = - \frac{1}{2}K + 0(K^3) \tag{23}$$

$$F = \left(\frac{3}{2\sqrt{2}} + 1 \right)K + 0(K^3) \tag{24}$$

$$H = F \tan \alpha = \frac{1}{6\sqrt{2}} \left(\frac{3}{2\sqrt{2}} + 1 \right) K^2 + 0(K^4) \tag{25}$$

Also from Eq. (20) we find the maximum height of the sheet is

$$y_{max} = y_o \Big|_{s = 1/2\sqrt{2}} K + \ldots = \frac{9}{512} K + 0(K^3) \tag{26}$$

Using Eqs. (9, 15) the tip of the cantilever is at

$$x(-1) = -1 + \left(\frac{1}{63} + \frac{1}{48\sqrt{2}} \right) K^2 + 0(K^4) \tag{27}$$

$$y(-1) = -\left(\frac{1}{8} + \frac{1}{6\sqrt{2}} \right) K + 0(K^3) \tag{28}$$

Numerical Integration

For general K the deflections are no longer small and numerical integration is necessary. Define

$$v = (\alpha, \lambda, F, k) \tag{29}$$

and let $x(s;v)$, $y(s;v)$, $\theta(s;v)$ be the solution to the initial value problem Eqs. (5, 8, 9) with the initial conditions Eqs. (6, 10, 29). Then the original two-point boundary value problem is equivalent to

$$f(v) = \left[y(k;v),\ \theta(k;v),\ \frac{d\theta}{ds}(k;v),\ \frac{d\theta}{ds}(-1;v)\right] = 0 \tag{30}$$

Equation (30) was solved by a combination of quasi-Newton and homotopy methods similar to that described in [1, 2]. The algorithm requires the Jacobian matrix $Df(v)$ of $f(v)$, and the partial derivatives $\frac{\partial f}{\partial v_i}(v)$. These are computed as follows:

Set $z_1 = x$, $z_2 = y$, $z_3 = \theta$, $z_4 = \dot{\theta} = \frac{d\theta}{ds}$, $z_5 = \frac{\partial x}{\partial v_i}$,

$z_6 = \frac{\partial y}{\partial v_i}$, $z_7 = \frac{\partial \theta}{\partial v_i}$, $z_8 = \frac{\partial \dot{\theta}}{\partial v_i}$ and consider the differential equation

$$\dot{z}_1 = \cos z_3$$

$$\dot{z}_2 = \sin z_3$$

$$\dot{z}_3 = z_4$$

$$\dot{z}_4 = -K(1 + s) \cos z_3 + F(\cos z_3 + \tan\alpha \sin z_3)$$

$$\dot{z}_5 = -z_7 \sin z_3$$

$$\dot{z}_6 = z_7 \cos z_3$$

$$\dot{z}_7 = z_8$$

$$\dot{z}_8 = K(1 + s)\, z_7 \sin z_3 + T \qquad (31)$$

where

$$T = \frac{\partial}{\partial v_i} (F (\cos z_3 + \tan\alpha \sin z_3)) \qquad (32)$$

has a different form depending on v_i. For $v_1 = \alpha$, the initial conditions are

$$z(0) = (0, 0, \alpha, \lambda, 0, 0, 1, 0); \qquad (33)$$

for $v_2 = \lambda$

$$z(0) = (0, 0, \alpha, \lambda, 0, 0, 0, 1); \qquad (34)$$

for $v_3 = F$

$$z(0) = (0, 0, \alpha, \lambda, 0, 0, 0, 0); \qquad (35)$$

for $v_4 = k$

$$z(0) = (0, 0, \alpha, \lambda, 0, 0, 0, 0). \qquad (36)$$

Thus solving the initial value problem given by Eqs. (31) and (33) produces, e.g., $\frac{\partial \dot{y}}{\partial \alpha}(k)$, which is the (1, 1) entry in the Jacobian matrix $Df(v)$. Using the differential Eq. (31) with $T = 0$ and initial conditions Eq. (33) or Eq. (34) produces the partials of $\dot{\theta}(-1)$, where the initial value problem is solved backwards from $s = 0$ to $s = -1$. Since the differential equation for $s \leq 0$ does not depend on F or k,

$$\frac{\partial \dot{\theta}}{\partial F}(-1) = \frac{\partial \dot{\theta}}{\partial k}(-1) = 0. \qquad (37)$$

These initial value problems were solved by a variable step, variable order ODE code [3] which is accurate, efficient, and robust. The combination of a quasi-Newton method [1], a globally convergent homotopy method [2, 4], and a sophisticated ODE method [3] proved to be very successful.

Results and Discussion

Fig. 2 shows the computed α, λ, F and H as a function of K. Also shown in the figure are our approximations for small K. All these parameters increase with K monotonically. Fig. 3 shows the geometric parameters y_{max}, $x(-1)$, $y(-1)$, x_c and k. As $K \to \infty$, all these parameters approach zero except $y(-1) \to 1$. Note that y_{max} is greatest $(= 0.4338)$ when $K = 8.25$. We keep in mind that all lengths have been normalized by the length of the cantilever segment.

Fig. 4 shows the shapes of the elastic sheet, for given overhang length ℓ, as ρ/EI is varied. Fig. 5 shows the situation when a given flexible sheet is gradually pushed off the corner (ρ/EI is fixed, while ℓ varies).

The present paper is related to the clamped cantilever studied by Bickley [5]. In his case, the governing equations are much simpler: Eqs. (5 - 7) with $\alpha = 0$. The single unknown λ may be obtained by shooting and does not require the quasi-Newton and homotopy methods used in this study. Bickley integrated the shape of the cantilever for $K < 14.51$. For the same K our problem shows larger deflection since the sheet is not clamped flat at the corner.

References

(1) J.E. Dennis and J.J. Moré, Quasi-Newton methods - motivation and theory. SIAM Review, 21, 46-79 (1977).

(2) L.T. Watson and C.Y. Wang, A homotopy method applied to elastica problems, Int. J. Solids Structures, 17, 29-37 (1981).

(3) L.F. Shampine and M.K. Gordon, Computer Solution of Ordinary Differential Equations: The Initial Value Problem, Freeman, San Francisco (1975).

(4) L.T. Watson, An algorithm that is globally convergent with probability one for a class of nonlinear two-point boundary value problems, SIAM J. Numer. Anal. 16, 394-401 (1979).

(5) W.G. Bickley, The heavy elastica, Phil. Mag. Ser. 7, 17, 603-622 (1934).

Figure Captions

Fig 1 The coordinate system.

Fig 2 Computed parameters as a function of K. Dashed lines are approximations.

Fig 3 Geometric parameters as a function of K.

Fig 4 Shapes of the elastic sheet for given overhang length and various K.

Fig 5 A given elastic sheet slowly pushed off a corner.

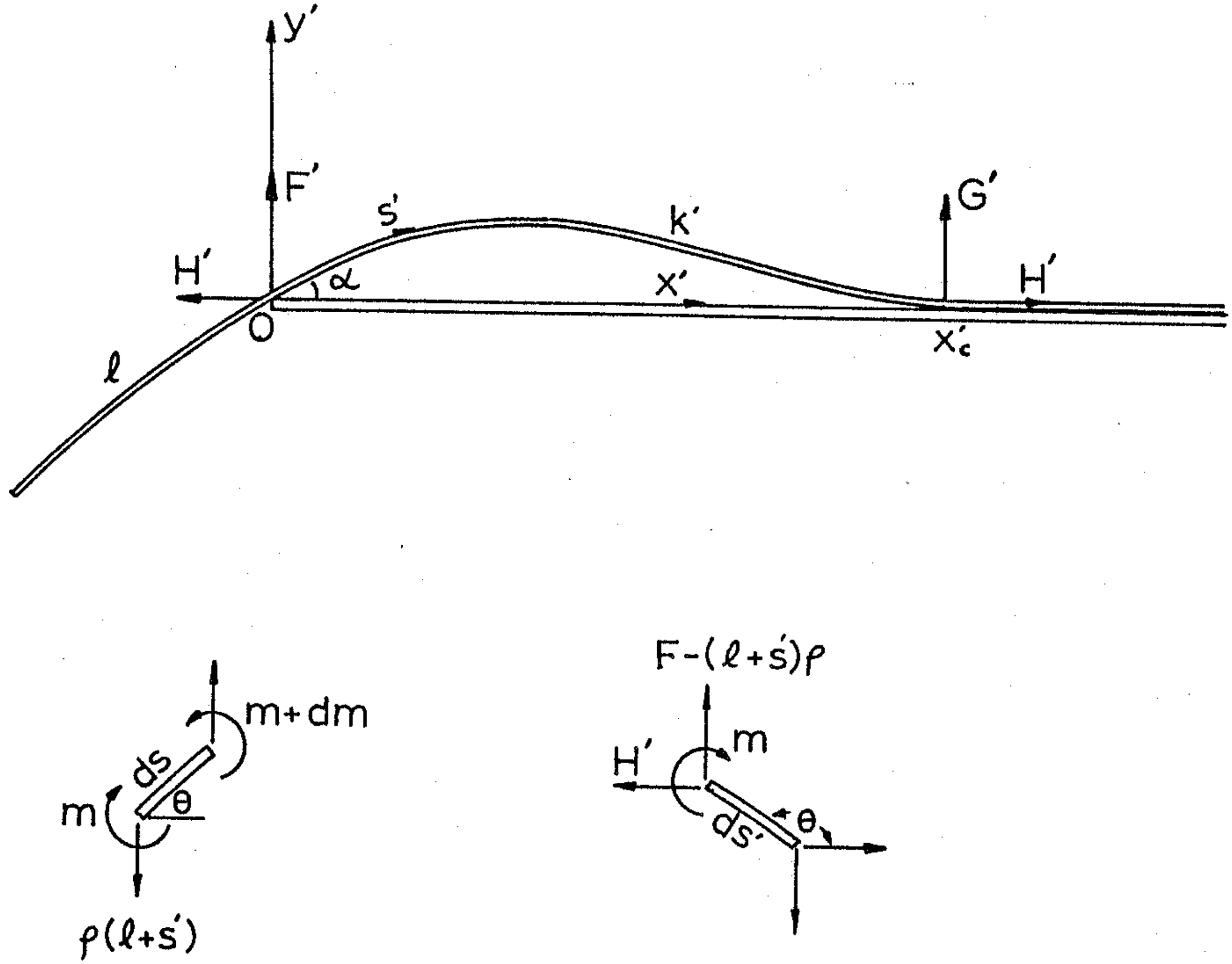

Fig 1 Watson Wang

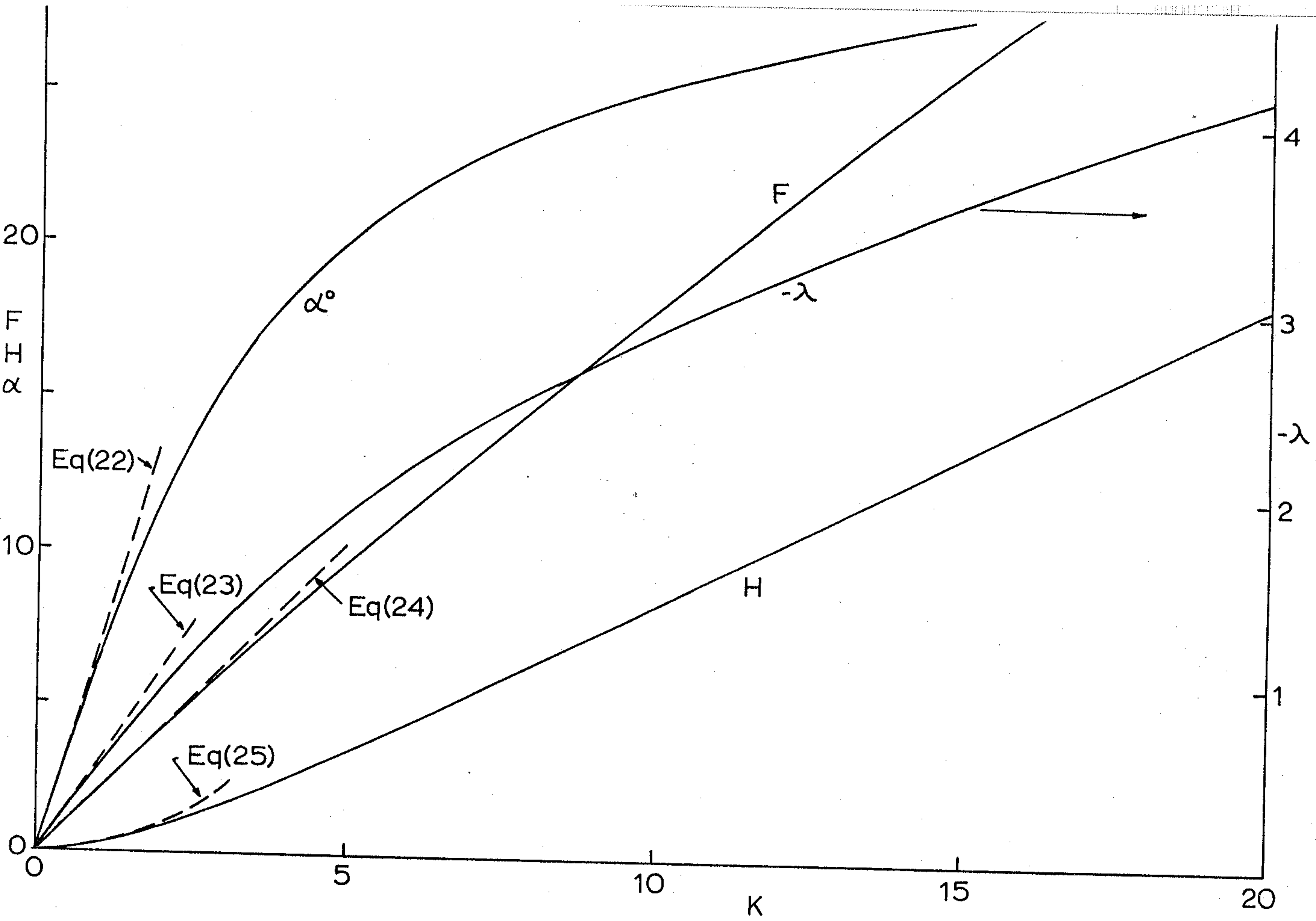

Fig 2 Watson-Wang

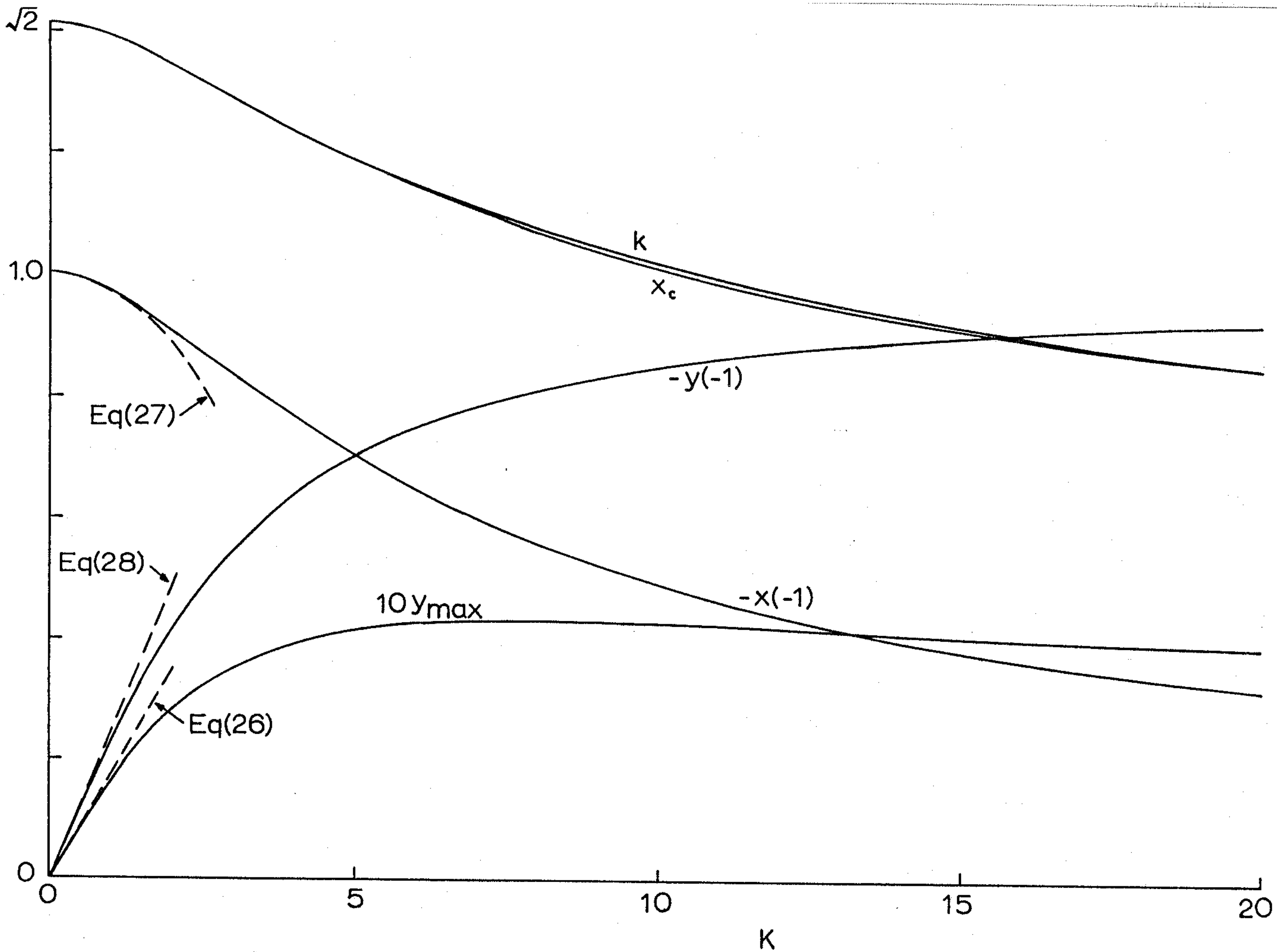

Fig 3 Watson Wang

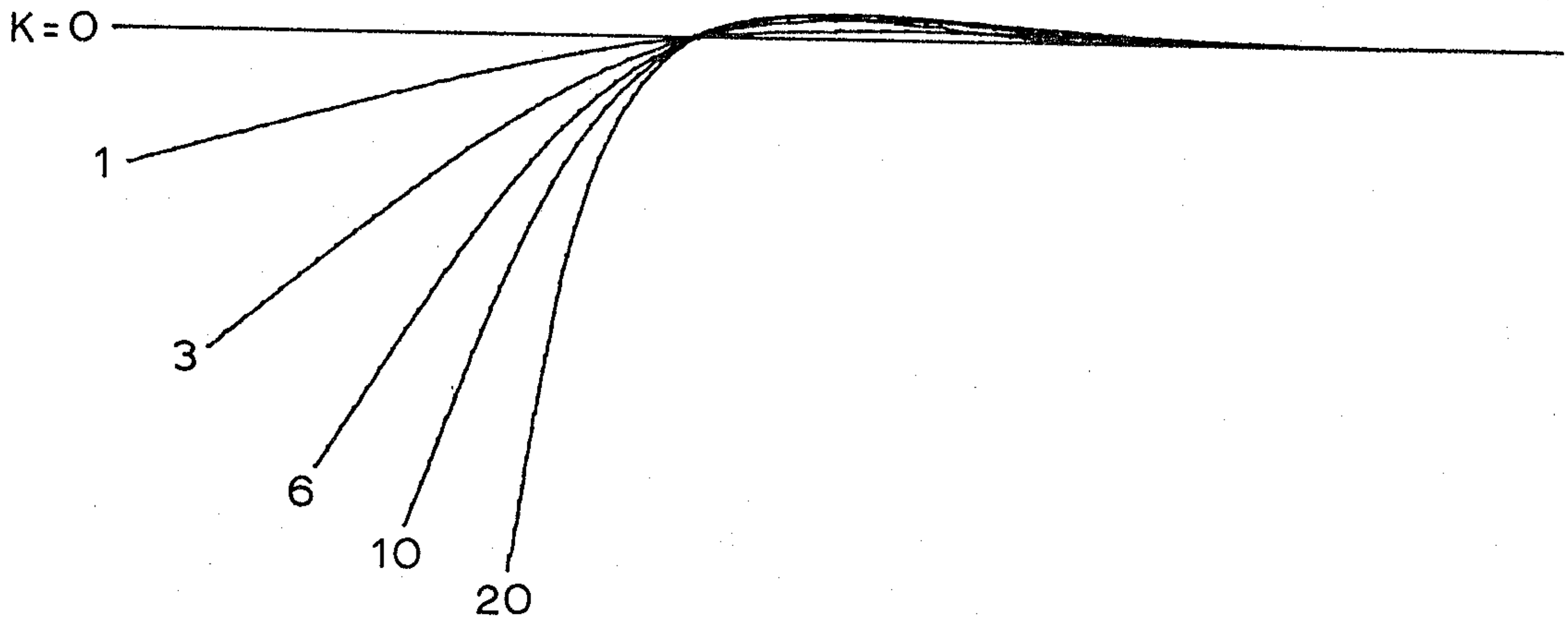

Fig 4 Watson-Wang

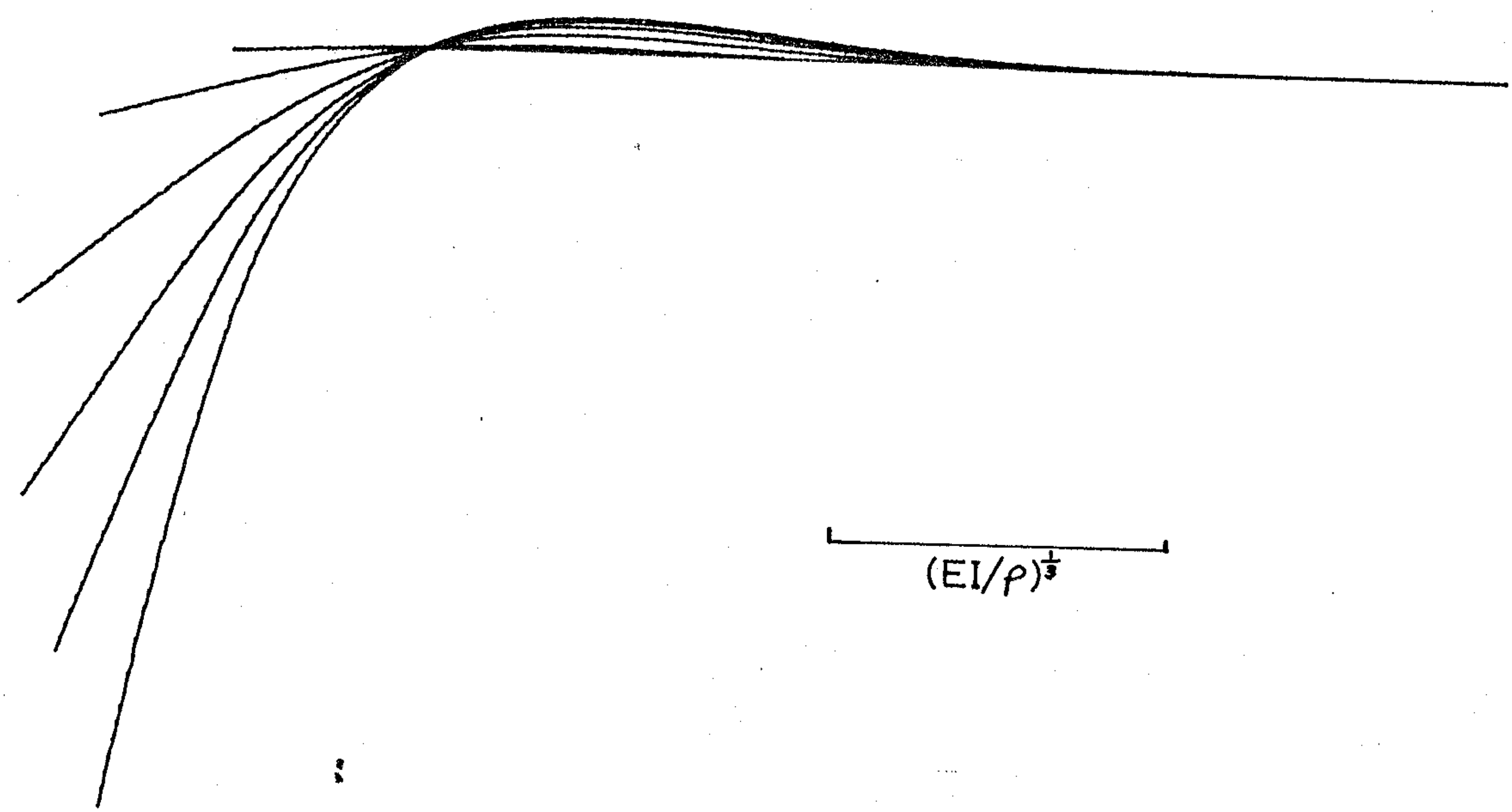

Fig 5 Watson-Wang