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# Engineering Applications of the Chow-Yorke Algorithm+

by

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Abstract. The Chow-Yorke algorithm is a scheme for developing homotopy methods that are globally convergent with probability one. Homotopy maps leading to globally convergent algorithms have been created for Brouwer fixed point problems, certain classes of nonlinear systems of equations, the nonlinear complementarity problem, some nonlinear two-point boundary value problems, and convex optimization problems. The Chow-Yorke algorithm has been successfully applied to a wide range of engineering problems, particularly those for which quasi-Newton and locally convergent iterative techniques are inadequate. Some of those engineering applications are surveyed here.

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## 1. Why homotopy methods?

A frequently asked and legitimate question is "Why do you need a homotopy method?" Just because a homotopy method is theoretically elegant and can be proven globally convergent does not justify its use if a simpler and more efficient method would suffice. The intent of this paper is to present a list of problems for which Newton and quasi-Newton methods are either totally inadequate or much more expensive than a globally convergent homotopy method.

As a simple example, consider the problem

$$t^2u - 1 = 0$$

$$u^2 - 1 = 0$$

This is a one-dimensional case of a structural design problem where t is the material thickness and u is the displacement. For this problem, Newton's method started from $(t,u) = (-2,-2)$ diverges. Very robust, well programmed quasi-Newton methods also fail. For example, least change secant update algorithms (sometimes erroneously called globally convergent), started at $(0,-1)$ fail because $(0,-1)$ is a local minimum for the norm of the function. This local minimum phenomenon is typical of fluid dynamics and elastica problems. Let

$$x = \begin{pmatrix} t \\ u \end{pmatrix}, \quad f(x) = \begin{pmatrix} t^2u-1 \\ u^2-1 \end{pmatrix}, \text{ and}$$

$$\rho(\lambda,x) = \lambda f(x) + (1-\lambda)(x-c).$$

Using the latter homotopy is also unsuccessful since the zero curve of $\rho(\lambda, x)$ does not reach $\lambda = 1$ (see Figure 1).

However, the homotopy map

$$\rho(\lambda,x) = f(x) - (1-\lambda)\begin{pmatrix} a \\ b \end{pmatrix}$$

does work. It is possible to prove that for almost all $\begin{pmatrix} a \\ b \end{pmatrix} \varepsilon E^1 \times (0,1)$ zero curves of $\rho(\lambda,x)$ reaching a solution exist [ 57 ]. See Figure 2. This example shows that there is probably not a "homotopy map for all seasons", but that some homotopy map, resulting in a globally convergent algorithm, may exist.

## 2. The Chow-Yorke Algorithm

The theoretical foundation of the Chow-Yorke algorithm is given in the following lemma [9,10,51]:

Def. Let U, $V \subset E^n$ be open sets and $\rho: U \times (0,1) \times V \to E^n$ be a $C^2$ map. $\rho$ is said to be transversal to zero if the Jacobian matrix $D\rho$ has full rank on $\rho^{-1}(0)$.

Parameterized Sard's Theorem. If $\rho(a,\lambda,x)$ is transversal to zero, then for almost all $a \;\varepsilon\; U$ the map

$$\rho_a(\lambda,x) = \rho(a,\lambda,x)$$

is also transversal to zero; i.e., with probability one the Jacobian matrix $D\rho_a(\lambda,x)$ has full rank on $\rho_a^{-1}(0)$.

The geometric interpretation of this result is that the set

$$\rho_a^{-1}(0) = \{(\lambda,x) | 0 \leq \lambda < 1, \rho_a(\lambda,x) = 0\}$$

of zeros of $\rho_a$ consists of smooth, disjoint curves which have no endpoints in $(0,1) \times V$ and have finite arc length in any compact subset of $(0,1) \times V$. This holds for almost all a, or, in other words, with probability one. See Figure 3.

The recipe for a globally convergent algorithm is then:

1) Construct a homotopy map $\rho(a,\lambda,x)$ such that

   a) $\rho$ is transversal to zero;

   b) $\rho_a(0,x) = 0$ is trivial to solve, and preferably has a unique solution;

   c) $\rho_a(1,x) = 0$ is equivalent to the given problem.

2) Prove that the zero curves of $\rho_a$ emanating from $\lambda = 0$ are bounded (and monotone in $\lambda$ if $\rho_a(0,x) = 0$ has more than one solution).

If 1) and 2) above have been accomplished, then for almost all $a$ there exists a zero curve $\nu$ of $\rho_a$, along which the Jacobian matrix $D\rho_a(\lambda,x)$ has full rank, emanating from $\lambda = 0$ and reaching a solution of the given problem at $\lambda = 1$ [10,51]. Thus a globally convergent algorithm consists of tracking this zero curve $\nu$ of $\rho_a$ from $\lambda = 0$ until it reaches $\lambda = 1$. The "Chow-Yorke algorithm" refers to 1), 2), and any scheme for tracking this zero curve $\nu$ of the homotopy map $\rho_a(\lambda,x)$.

There is some controversy over how to track this zero curve $\nu$. A scheme is summarized here (see [51,52 ] for more details) which the author has found to be accurate, easy to use, reliable, robust, and efficient for practical problems. Since the zero curve $\nu$ is smooth, it can be parameterized by arc length $s$. Thus $\lambda = \lambda(s)$, $x = x(s)$ along $\nu$ and

$$\rho_a(\lambda(s), x(s)) = 0 \qquad (1)$$

identically in $s$. Let $\nu$ emanate from $(0,x_0)$. Then $\nu$ is the trajectory of the initial value problem

$$\frac{d}{ds}\,\rho_a(\lambda(s),\ x(s)) = D\rho_a(\lambda(s),\ x(s)) \begin{pmatrix} \frac{d\lambda}{ds} \\ \frac{dx}{ds} \end{pmatrix} = 0, \tag{2}$$

$$\left\| \begin{pmatrix} \frac{d\lambda}{ds} \\ \frac{dx}{ds} \end{pmatrix} \right\|_2 = 1, \tag{3}$$

$$\lambda(0) = 0,\ x(0) = x_0 \quad . \tag{4}$$

Recall that (for almost all a) the Jacobian matrix

$$D\rho_a(\lambda(s),\ x(s)) \tag{5}$$

has full rank. Therefore, (5) has a one-dimensional kernel, and $(d\lambda/ds,\ dx/ds)$ is uniquely determined by (2), (3), and continuity. The kernel of the matrix (5) is determined in a numerically stable and accurate way by factoring (5) with Householder reflections [7,51,56 ]. Values of $(d\lambda/ds,\ dx/ds)$ are used as input to an ODE solver which solves the initial value problem (2-4). Since evaluation and factorization of the Jacobian matrix (5) is expensive, an ODE solver which puts a premium on minimizing the number of derivative evaluations seems appropriate. For example, the subroutines STEP and INTRP of [ 42 ] work very well in this context. For some practical considerations regarding the tracking of $\nu$ and obtaining the solution at $\lambda = 1$, see [ 51 ], [ 52 ], [ 55 ], and [ 56 ].

3) Engineering Applications

To give some idea of how widely applicable the Chow-Yorke algorithm is, a partial list of problems solved by the Chow-Yorke algorithm is presented. These problems range from fairly simple to extremely difficult, and Newton-type methods either partially or totally failed on all of them.

1. Elliptic porous slider.
2. Squeezing of a viscous fluid between parallel plates.
3. Squeezing of a viscous fluid between elliptic plates.
4. Viscous flow between rotating discs with injection on the porous disc.
5. Deceleration of a rotating disc in a viscous fluid.
6. Porous channel flow in a rotating system.
7. Optimal structural design (continuum mechanics).
8. Convex unconstrained optimization.
9. Optimization with nonnegativity constraints.
10. Nonlinear complementarity problem.
11. Large deformation of an elastic rod.
12. Large deformation of C-clamps.
13. Large deformation of negator clips.
14. Fluid-filled cylindrical membrane container.
15. Circular leaf spring.
16. Hanging elastic ring.
17. Equilibrium of heavy elastic cylindrical shells.
18. Equilibrium of reticulated shells.
19. Collapse of tethered blood vessels.

A few of these will now be discussed in more detail.

## ELLIPTIC POROUS SLIDER

Consider an air-cushioned vehicle, supported by air-pressue from air forced down through its base, with an elliptic base. The important quantities are lift, drag, and the most efficient direction in which to move the vehicle. The fluid flow is described by the nondimensional equations [ 54 ]:

$$R[(h')^2 - (h + k)h''] = Q + h'''$$

$$R[(k')^2 - (h + k)k''] = \beta^2 Q + k'''$$

$$R[fh' - (h + k)f'] = f''$$

$$R[gk' - (h + k)g'] = g''$$

$$h(0) = k(0) = h'(0) = k'(0) = h'(1) = k'(1) = 0$$

$$h(1) + k(1) = f(0) = g(0) = 1, \; f(1) = g(1) = 0$$

where $\beta$ is the eccentricity of the elliptic base and f, g, h, k represent velocities and pressures in some coordinate system. Let

$$v = \begin{pmatrix} f'(0) \\ g'(0) \\ h''(0) \\ k''(0) \\ Q \end{pmatrix}, \quad F(v) = \begin{pmatrix} f(1) \\ g(1) \\ h(1) + k(1) - 1 \\ h'(1) \\ k'(1) \end{pmatrix}, \quad \text{and}$$

$\rho_a(\lambda, v) = \lambda F(v) + (1 - \lambda)(v - a)$ be the homotopy used to solve $F(v) = 0$, which is equivalent to the two-point boundary value problem. This approach worked very well. An interesting result is that the most efficient direction in which to operate the slider is along its minor axis, i.e., sideways.

## SQUEEZING OF A VISCOUS FLUID BETWEEN PARALLEL PLATES

The governing equations are [ 44 ]:

$$S(nf''' + 3f'' + m f'f'' - ff''') = f^{(4)}, \quad m = 0,1$$

$$f(0) = f''(0) = 0, \; f(1) = 1, \; f'(1) = 0$$

This problem is similar to but much simpler than the next problem.

## SQUEEZING OF A VISCOUS FLUID BETWEEN ELLIPTIC PLATES

The governing (nondimensional) equations are [ 45 ]:

$$f''' + K = S[2f' + \eta f'' + \frac{1}{2} f'f' - \frac{1}{2} f'' (f + g)]$$

$$g''' + \beta K = S[2g' + \eta g'' + \frac{1}{2} g'g' - \frac{1}{2} g'' (f + g)]$$

$$f(0) = g(0) = f''(0) = g''(0) = f'(1) = g'(1) = 0,$$

$$f(1) + g(1) = 2$$

where $\beta$ is the eccentricity of the ellipses, $S$ is a Reynolds number, $f$ and $g$ describe the flow, and K is a constant to be determined. $v$, $F(v)$, and $\rho_a(\lambda,v)$ are defined analagously to the elliptic porous slider problem. This problem displays extreme sensitivity for $S > 20$, and very complicated behavior for $S < 0$. Figure 4 shows the complicated geometry of the solution surfaces for a particular set of the parameters (note the multiple solutions and catastrophe at $\beta = 1$).

## POROUS CHANNEL FLOW IN A ROTATING SYSTEM

Lubrication in rotating machinery and flow under the polar ice cap are examples of porous channel flow in a rotating system. The nondimensional governing equations are [ 60 ]:

$$R(f'f'' - ff''') = f^{(4)} + \nu k',$$

$$R(f'k - fk') = k'' - \nu f',$$

$$R(gf' - fg') = g'' + \nu h + B,$$

$$R(gk - fh') = h'' - \nu g,$$

$$f(0) = -1,\ f(1) = -\beta,\ f'(0) = f'(1) = 0,$$

$$k(0) = k(1) = g(0) = g(1) = h(0) = h(1) = 0.$$

f, g, h, and k describe the flow and $\nu$, R, B, $\beta$ are parameters. There are boundary layers at both 0 and 1 as well as internal boundary layers, which makes this problem extremely difficult. For $\nu$ and R small, the homotopy

$$\rho_a(\lambda, v) = \lambda F(v) + (1 - \lambda)(v - a) \tag{6}$$

with F(v) defined by shooting was adequate to solve the problem. Newton and quasi-Newton methods were completely inadequate for this problem. For $\nu$, $R \geq 30$, $\beta < 0$, $B = .5$ shooting becomes impossible because of the sensitivity of the problem, and F(v) defined by a finite difference approximation of the two-point boundary value problem was used in the homotopy (6). This approach was quite successful [ 61 ], although the resulting F(v) is a high dimensional nonlinear function.

## OPTIMAL STRUCTURAL DESIGN

A class of problems in optimal structural design have the form

$$K(t)u = f \tag{7}$$

$$u^t B_i u = 1, \; i = 1, \ldots , m$$

where

$$K(t) = \sum_{i=1}^{m} t_i^2 K_i,$$

the $K_i$ are $n \times n$ positive semidefinite matrices, K(t) is positive definite if $t > 0$, the $B_i$ are positive semidefinite matrices, and $m < n$. $t_i^2$ is the thickness of the ith element, and $u_j$ is the displacement of the jth node in the structure. Given a load vector f, the problem is to find the material thicknesses $t_i$ and the nodal displacements $u_j$ such that the energy density is uniform.

Quasi-Newton methods applied to (7) frequently fail because the norm of the function has many local minima. Also, the homotopy (6) virtually always fails with the zero curves going off to infinity, where

$$v = \begin{pmatrix} t \\ u \end{pmatrix}, \quad F(v) = \begin{pmatrix} K(t)u - f \\ u^t B_1 u - 1 \\ \cdot \\ \cdot \\ \cdot \\ u^t B_m u - 1 \end{pmatrix} \qquad (8)$$

In [ 57 ] it was proved that there is a globally convergent homotopy method for (7) constructed as follows. Define

$\psi: E^n \times (0, 1)^m \times [0, 1) \times E^m \times E^n \rightarrow E^{m+n}$ by

$\psi(a, b, \lambda, t, u) =$

$$\begin{pmatrix} [\lambda K(t) + (1 - \lambda) \operatorname{diag}(t_1^2, \ldots, t_m^2, 1, \ldots, 1)]u - f - (1 - \lambda)a \\ u^t[\lambda B_1 + (1 - \lambda)e_1 e_1^t]u - 1 - (1 - \lambda)b_1 \\ \vdots \\ u^t[\lambda B_m + (1 - \lambda)e_m e_m^t]u - 1 - (1 - \lambda)b_m \end{pmatrix}$$

where $e_i$ is the ith standard basis vector in $E^n$. Now regard $\psi$ as a <u>complex</u> map

$\tilde{\psi}: C^n \times C^m \times [0, 1) \times C^m \times C^n \rightarrow C^{m+n}$

which in turn may be regarded as a <u>real</u> map

$\rho: E^{2n+2m} \times [0, 1) \times E^{2m+2n} \rightarrow E^{2m+2n}$ .

The homotopy actually used is

$$\rho_d(\lambda, x) = \rho(d, \lambda, x), \qquad (9)$$

and the theorem is that for almost all $d$ the zero curves of $\rho_d$ are monotone and bounded, and therefore reach a solution of (7) at $\lambda = 1$.

An interesting observation is that the quasi-Newton method applied to the complexification of (8),

$$G(x) = 0 \tag{10}$$

obtained by converting (8) to complex form and then back to real again, always worked. Intuitively, $\| G(x) \|$ does not have the same local minima that $\| F(v) \|$ does. Undoubtedly some general theorems must hold regarding the effect of such "complexification".

## OPTIMIZATION

Given the generality of theorems for global convergence of the Chow-Yorke algorithm, it is not surprising that globally convergent homotopy algorithms exist for some optimization problems. Of course it is debatable whether homotopy methods are competitive with existing optimization techniques, but homotopy methods are not a priori worse. Some sample theorems [ 58 ] are presented here. Consider the problem

$$\min_x f(x) \tag{11}$$

Theorem. Let $f: E^n \to E$ be a $C^3$ convex map with a minimum at $\tilde{x}$, $\|\tilde{x}\| \leq M$. Then for almost all $a$, $\|a\| < M$, there exists a zero curve $\gamma$ of the homotopy map

$$\rho_a(\lambda, x) = \lambda \nabla f(x) + (1 - \lambda)(x - a) ,$$

along which the Jacobian matrix $D\rho_a$ has full rank, connecting $(0,a)$ to $(1, \bar{x})$, where $\bar{x}$ solves (11).

Say that $f$ is <u>uniformly convex</u> if its Hessian's smallest eigenvalue is bounded away from zero. Consider the constrained problem

$$\min f(x) \quad \text{such that } x \geqq 0 \tag{12}$$

<u>Theorem</u>. Let $f: E^n \to E$ be a $C^3$ uniformly convex map. Then there exists $\delta > 0$ such that for almost all $a \geqq 0$ with $\|a\| < \delta$ there exists a zero curve $\gamma$ of the homotopy map

$$\rho_a(\lambda, x) = \lambda K(x) + (1 - \lambda)(x - a),$$

where

$$K_i(x) = -\left|\frac{\partial f(x)}{\partial x_i} - x_i\right|^3 + \left(\frac{\partial f(x)}{\partial x_i}\right)^3 + x_i^3 ,$$

along which the Jacobian matrix $D\rho_a$ has full rank, connecting $(0,a)$ to $(1, \bar{x})$, where $\bar{x}$ solves (12).

## NONLINEAR COMPLEMENTARITY

Complementarity problems, both linear and nonlinear, are important and very widely studied. Given $F: E^n \to E^n$, the nonlinear complementarity problem is to find an $x \in E^n$ such that

$$x \geqq 0, \quad F(x) \geqq 0 , \qquad x^t F(x) = 0 \tag{13}$$

Since one normally thinks of homotopy methods as solving nonlinear systems of equations or computing fixed points, it is interesting that this problem with inequalities can be solved. Several theorems which prove the existence of a globally convergent homotopy method for (13) will be stated. Some computational experience with these homotopies is in [53,58]. Define $G: E^n \to E^n$ by

$$G_i(z) = -|F_i(z) - z_i|^3 + (F_i(z))^3 + z_i^3$$

Theorem (Mangasarian). $z$ solves (13) if and only if

$$G(z) = 0 \tag{14}$$

The homotopy used is

$$\rho_a(\lambda,z) = \lambda G(z) + (1 - \lambda)(z - a). \tag{15}$$

Theorem. Let $F:E^n \to E^n$ be a $C^2$ map, and let the Jacobian matrix $DG(z)$ be nonsingular at every zero of $G(z)$. Suppose there exists $r > 0$ such that $z > 0$ and $z_k = \|z\|_\infty \geqq r$ imply $F_k(z) > 0$. Then for almost all $a > 0$ there exists a zero curve $\nu$ of $\rho_a(\lambda,z)$, on which the Jacobian matrix $D\rho_a(\lambda,z)$ has full rank, having finite arc length and connecting $(0,a)$ to $(1,\bar{z})$, where $\bar{z}$ solves (13).

Theorem. Let $F:E^n \to E^n$ be a $C^2$ map, and let the Jacobian matrix $DG(z)$ be nonsingular at every zero of $G(z)$. Suppose there exists $r > 0$ such that $z \geqq 0$ and $\|z\|_\infty \geqq r$ imply $z_k F_k(z) > 0$ for some index $k$. Then there exists $\delta > 0$ such that for almost all $a \geqq 0$ with $\|a\|_\infty < \delta$ there exists a zero curve $\nu$ of $\rho_a(\lambda,z)$, along which the Jacobian matrix $D\rho_a(\lambda,z)$ has full rank, having finite arc length and connecting $(0,a)$ to $(1,\bar{z})$, where $\bar{z}$ solves (13).

These two theorems subsume most linear cases $(F(z) = Mz + q)$ of interest. Note that such homotopy theorems simultaneously prove existence and provide an algorithm for calculating the solution.

## ELASTIC ROD

Consider a thin incompressible elastic rod clamped at the origin and acted on by forces Q, P and torque M (see Figure 5). The governing (non-dimensional) equations are:

$$\frac{dx}{ds} = \cos\theta , \quad \frac{dy}{ds} = \sin\theta , \quad \frac{d\theta}{ds} = Qx - Py + M \tag{16}$$

$$x(0) = y(0) = \theta(0) = 0 \tag{17}$$

$$x(1) = a, \quad y(1) = b, \quad \theta(1) = c \tag{18}$$

The cantilever beam problem, which has a closed form solution in terms of elliptic integrals, is to find the position (a,b) of the tip of the rod given the forces $Q \neq 0$ and $P = 0$. Consider the inverse problem, where the a, b, c are specified, and Q, P, M are to be determined. For large c, $c = 10\pi$ for example, the elastica is wound like a coil spring and its shape is extremely sensitive to small perturbations in Q, P, or M. For large deformations the problem (16-18) is ferociously nonlinear, and Newton and quasi-Newton methods generally fail [ 63 ].

The Chow-Yorke algorithm was completely successful on (16-18) using the homotopy map

$$\psi(d,\lambda,v) = \begin{pmatrix} x(1;v) - [\lambda a + (1-\lambda)d_1] \\ y(1;v) - [\lambda b + (1-\lambda)d_2] \\ \theta(1;v) - [\lambda c + (1-\lambda)d_3] \end{pmatrix}$$

where $v = \begin{pmatrix} Q \\ P \\ M \end{pmatrix}$ and $x(s;v)$, $y(s;v)$, $\theta(s;v)$ are the solution to the initial value problem (16-17). In [ 63 ] numerous approaches to this inverse elastica problem were considered, with a homotopy method using the above homotopy map being the most successful. The homotopy

$$\rho_a(\lambda, v) = \lambda F(v) + (1 - \lambda)(v - a)$$

with $v = \begin{pmatrix} Q \\ P \\ M \end{pmatrix}$, $F(v) = \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \begin{pmatrix} x(1;v) - a \\ y(1;v) - b \\ \theta(1;v) - c \end{pmatrix}$

was unsuccessful on this problem for every sign combination.

## C-CLAMP

Consider an elastic C-shaped clamp with natural curvature $M_0$ as shown in Figure 6. The governing equations are similar to those of the elastic rod, but the boundary conditions are different. The equations are:

$$\frac{dx}{ds} = \cos\theta\ , \quad \frac{dy}{ds} = \sin\theta\ , \quad \frac{d\theta}{ds} = -Fy + M_1 + M_0\ ,$$

$$x(0) = y(0) = \theta(0) = 0\ ,$$

$$x(1) = a,\ \frac{d\theta}{ds}(1) = M_0 \qquad (M_0 = \text{natural curvature}).$$

The solution details are similar to the elastic rod, and need not be repeated. For a complete discussion, see [ 47 ].

## NEGATOR CLIP

A related problem involves the negator clip or so-called "constant force spring". A spring with natural curvature $M_0$ is wound into two coils of equal length (see Figure 7). It has been claimed [ 48 ] that the force exerted by the separated coils is independent of the separation of the coils. This is in fact true asymptotically, but the force F is a nonlinear function of lateral displacement $x(L)$, where $L$ is the arc length OA of the unwound spring, for moderate L/R ratios, where $R$ is the natural radius of the spring. The governing equations are:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta, \quad \frac{d\theta}{ds} = M_0 - M_1 + F_y,$$

$$x(0) = y(0) = \theta(0) = 0$$

$$\theta(L/R) = \pi/2, \quad Fy(L/R) - M_1 = 0,$$

where $M_0$ is the natural curvature, $M_1$ is the maximum moment occurring at the point of symmetry, and $s$ is a nondimensional variable. For a complete discussion see [ 48 ].

## LEAF SPRING

Another, but much more difficult, spring problem is the leaf spring [ 64 ] (Figure 8). The governing equations

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta, \quad \frac{d\theta}{ds} = M_0 - M + Fx$$

$$x(0) = y(0) = \theta(0) = 0$$

$$Fx(1) - M = 0, \quad y(1) = b \qquad (M_0 = \text{natural curvature})$$

are very similar to those for the negator clip, but there are multiple solutions, turning points, and bifurcation points as shown in Figure 8. The Chow-Yorke algorithm is not designed to handle bifurcation points, and the bifurcation point shown in Figure 8 was obtained by trial and error. The homotopy maps for all of these elastica problems are similar to the elastic rod homotopy. See [ 64 ] for a complete discussion of the leaf spring problem.

## FLUID-FILLED CYLINDRICAL MEMBRANE CONTAINER

A rather different kind of elastica problem concerns a membrane container filled with a fluid. Depending on the rigidity of the container wall and the internal fluid pressure, the container sags making contact with the ground (Figure 9). For low pressures and rigidity, the cross-sectional shape is oblong and the container has a small volume compared to a circular cylinder. For high internal pressures or very rigid material the shape is almost circular. The interesting question is the trade off between pressure and volume, since it is difficult and expensive to obtain high internal pressures, yet low pressures waste container material since the volume is comparatively small. The (nondimensional) governing equations are:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta, \quad \frac{d\theta}{ds} = \frac{1}{\alpha}(\beta - y)$$

$$x(0) = y(0) = \theta(0) = 0$$

$$x(1 - c) = -c, \quad y(1 - c) = 0, \quad \theta(1 - c) = 2\pi,$$

where $\beta$ is a given constant, $c$ is the unknown contact length, and $\alpha$ is a parameter to be determined. What makes this problem different from the previous ones is that the interval of integration $1 - c$ is unknown, and the boundary condition

$$x(1 - c) = -c$$

is difficult to handle. Nevertheless, a straightforward homotopy was successful [ 49 ].

## HEAVY ELASTIC CYLINDER

Important construction problems in outer space and undersea involve heavy elastic cylinders. Depending on the rigidity of the elastic wall material, the cylinder may collapse under its own weight. There are four distinct cases, governed by a nondimensional parameter $B$ (see Figure 10). Starting from a perfect cylinder $(B = 0)$, as $B$ increases the point contact (Case 1) widens to a line contact (Case 2) then the top sags until it touches the bottom for a point-line contact (Case 3), then ultimately the top also makes a line contact with the bottom. The governing equations for all four cases are

$$\frac{dx}{ds} = \cos\theta , \quad \frac{dy}{ds} = \sin\theta ,$$

$$\frac{d^2\theta}{ds^2} = A \sin\theta + (C - Bs)\cos\theta .$$

For Case 1, $C = B$ and the boundary conditions are

$$x(0) = y(0) = \theta(0) = 0 ,$$

$$x(1) = 0 , \quad \theta(1) = \pi .$$

For Case 2, $C = B(1 - a)$ and the boundary conditions are

$$x(0) = y(0) = \theta(0) = \dot{\theta}(0) = 0 ,$$

$$x(1 - a) = -a, \quad \theta(1 - a) = \pi .$$

For Case 3, the boundary conditions are

$$x(0) = y(0) = \theta(0) = \dot{\theta}(0) = 0 ,$$

$$x(1 - a) = -a, \quad y(1 - a) = 0, \quad \theta(1 - a) = \pi .$$

For Case 4, the boundary conditions are

$$x(0) = y(0) = \theta(0) = \dot{\theta}(0) = 0 ,$$

$$y(b) = 0 , \quad \dot{\theta}(b) = 0 .$$

For Cases 1 and 2, quasi-Newton methods are adequate and efficient if a good computer code is used. For Cases 3 and 4, where B is large, quasi-Newton methods are feasible but very expensive because of their small domain of practical application. If the starting point is too far away from the solution, quasi-Newton codes such as HYBRJ from Argonne's MINPACK fail to make progress toward the solution and give an error return [ 50 ]. The homotopy map

$$\rho_a(\lambda, v) = \lambda F(v) + (1 - \lambda)(v - a) ,$$

where $v$ consists of the appropriate initial conditions and parameters (depending on the case) and $F(v)$ is defined by shooting, works very well for large B [ 50 ]. This is a rare example of a problem on which quasi-Newton methods do not totally fail, and yet the homotopy algorithm is more efficient. Generally, quasi-Newton methods, when they work, are an order of magnitude more efficient than homotopy methods.

## 4) Conclusion

Differential geometry provides a solid theoretical foundation for the Chow-Yorke algorithm [ 2,3 ], and homotopy maps producing globally convergent algorithms have been constructed for a wide range of problems. Perhaps the most spectacular successes have been for Brouwer fixed points [ 51 ] and the nonlinear complementarity problem [ 53 ]. The numerous engineering problems discussed here show that homotopy methods are frequently successful on problems to which the (known) theory is not applicable. The prospect of a globally convergent algorithm, particularly on problems for which the best quasi-Newton computer code [ 37 ] fails, makes homotopy methods appealing and promising for future development.

On the negative side, the supporting differential geometry theory requires at least $C^2$ smoothness, which means the Chow-Yorke algorithm cannot handle directly, e.g., piecewise linear maps (see [ 4 ], though). Also, developing a homotopy map whose zero curves are bounded is very difficult, and, at present, an art. Finally, homotopy methods are computationally expensive (at least an order of magnitude worse than quasi-Newton methods), and there is general agreement that they should only be used as a last resort.

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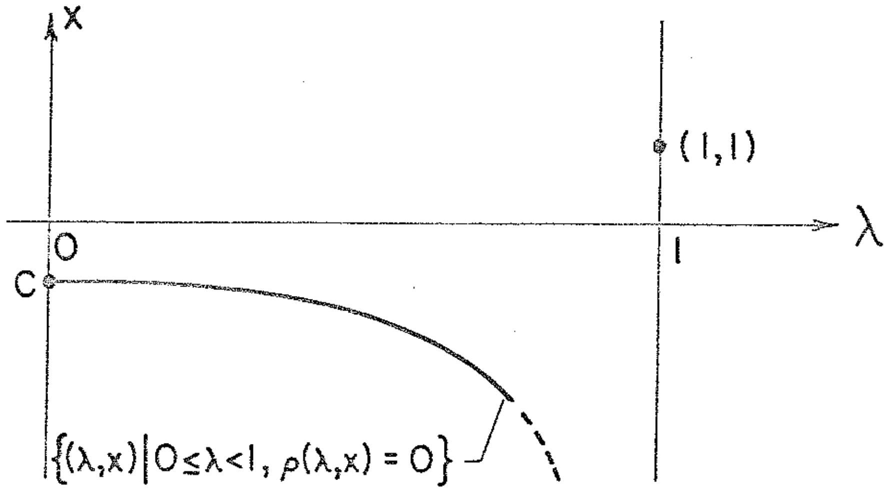


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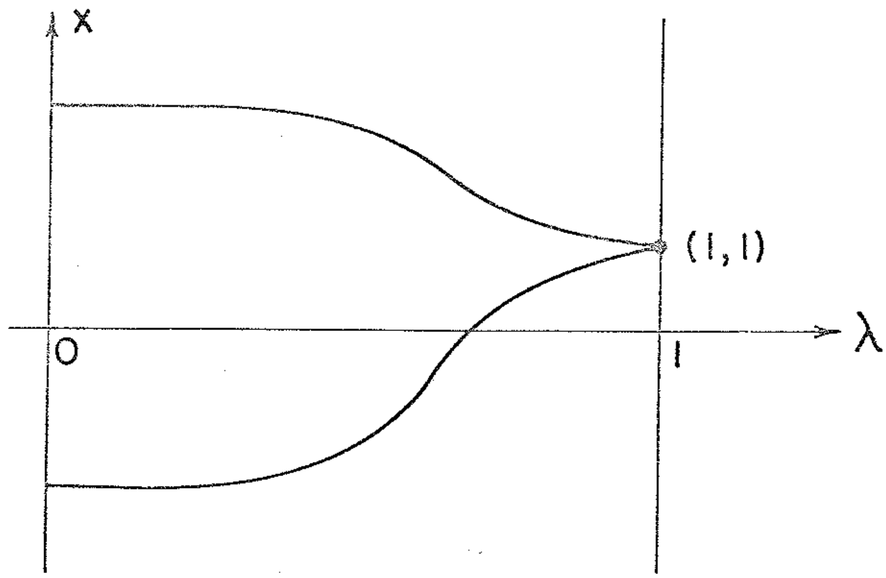

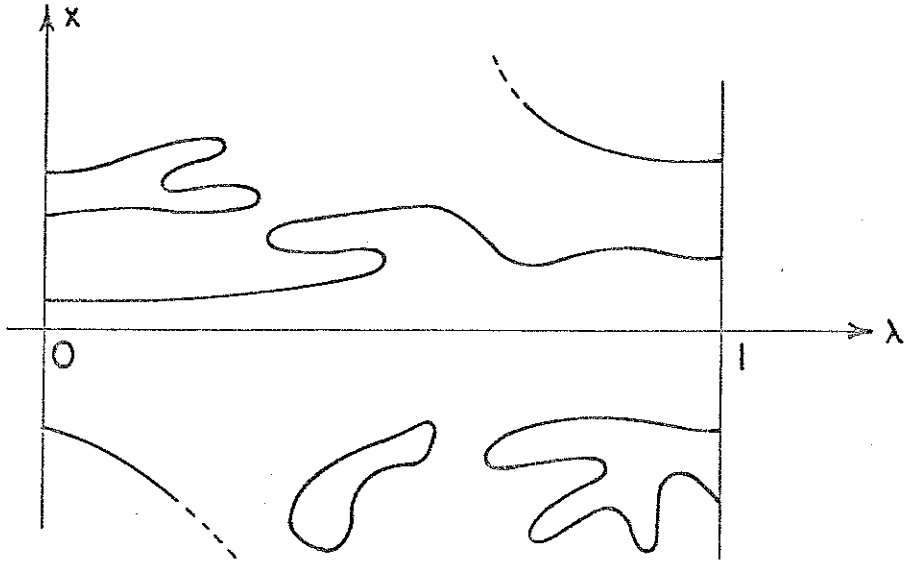


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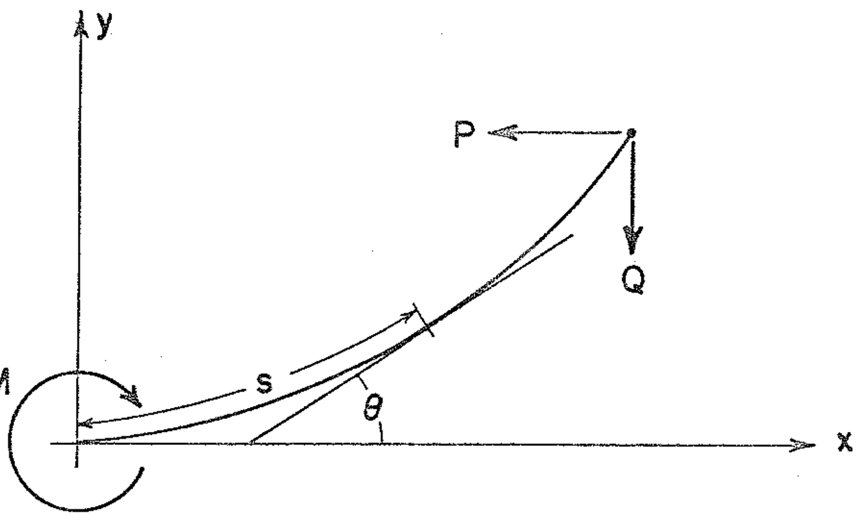
Figure




Figure
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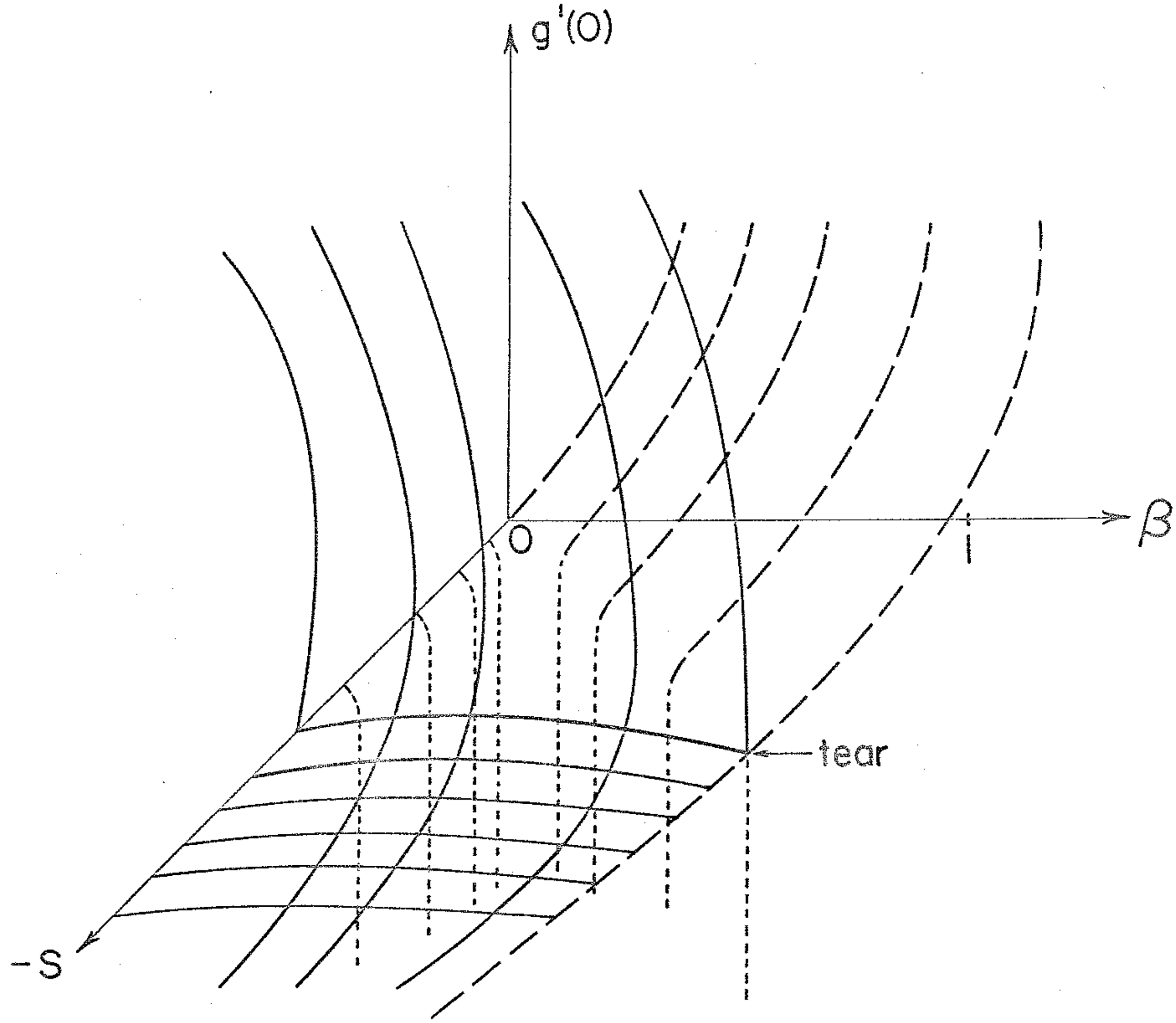
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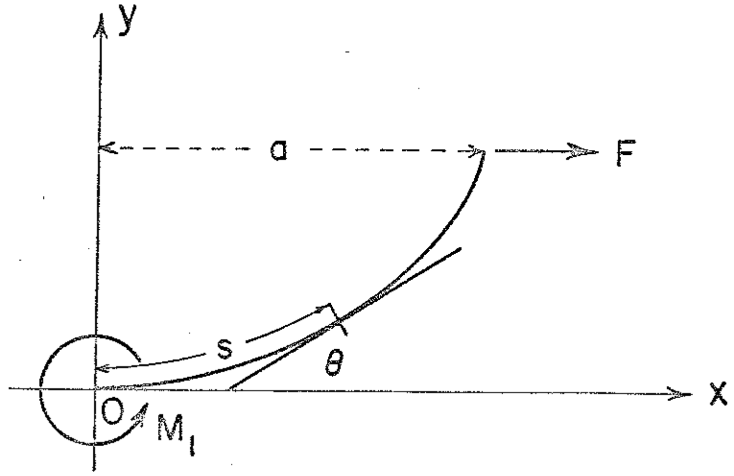


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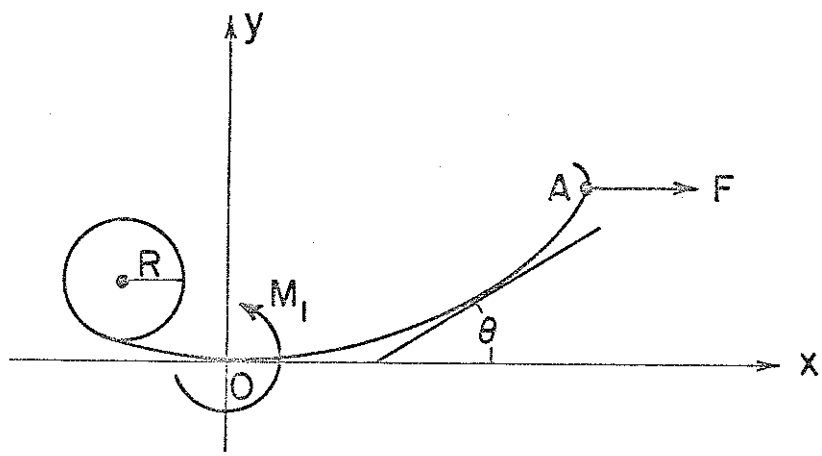


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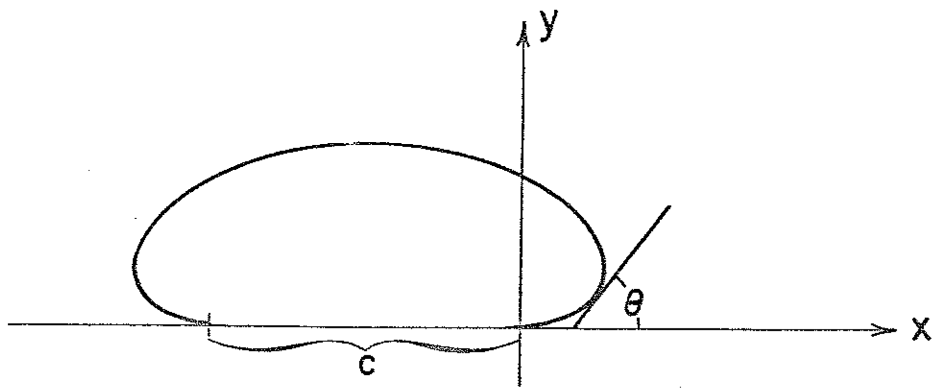


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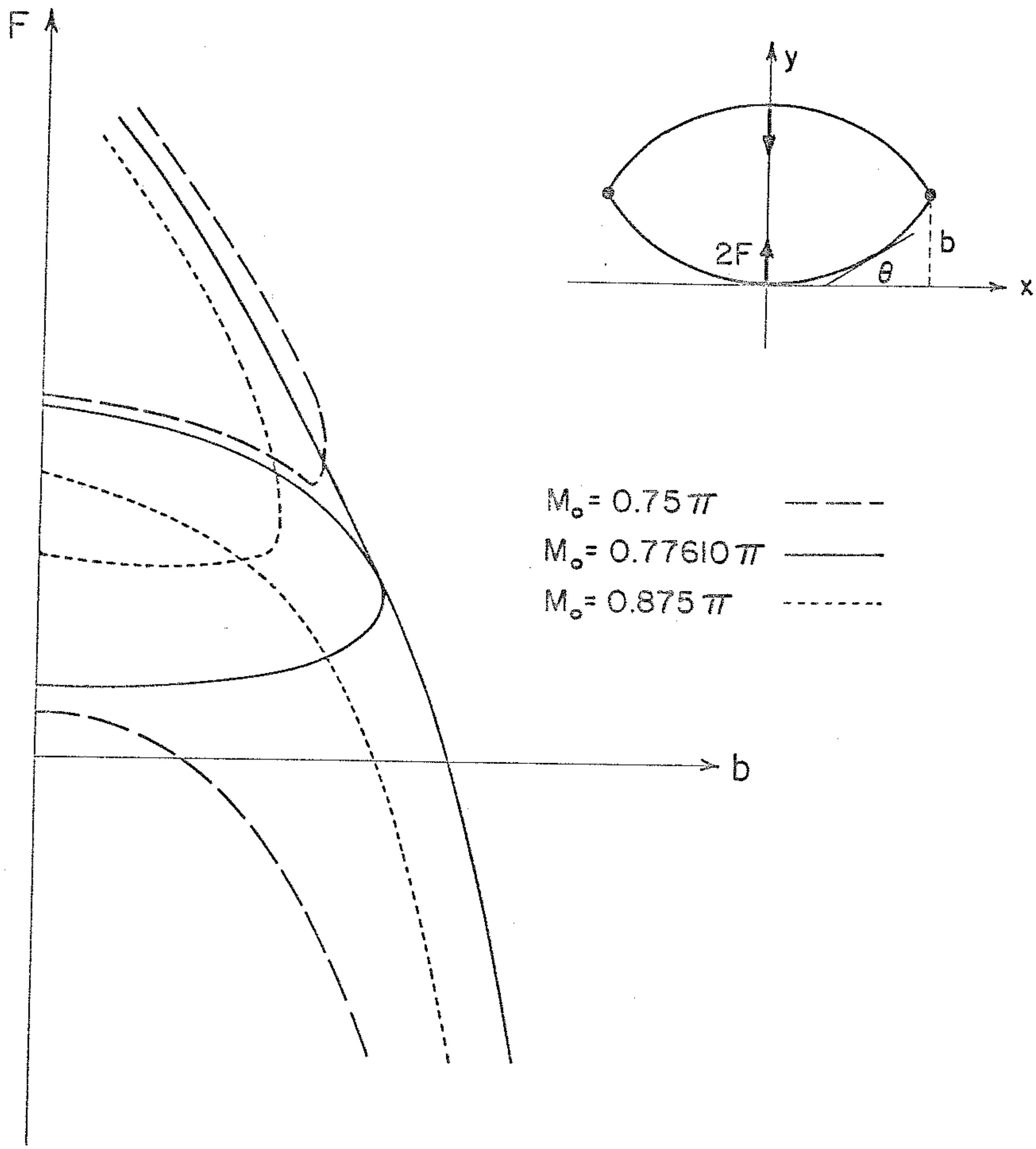
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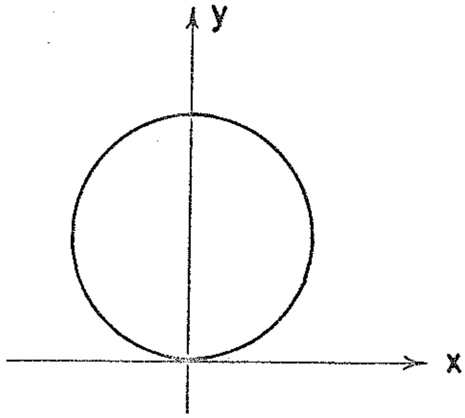
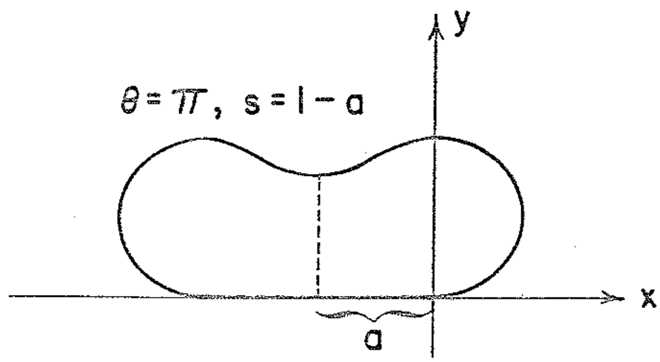



CASE 1 (POINT CONTACT)

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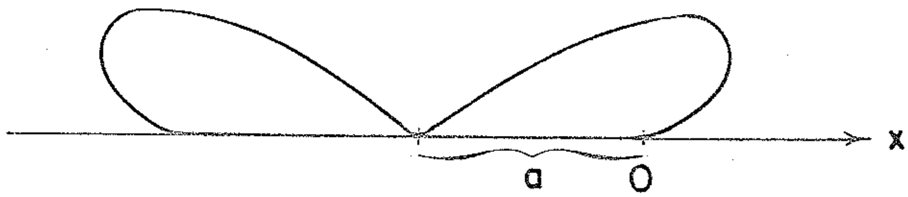
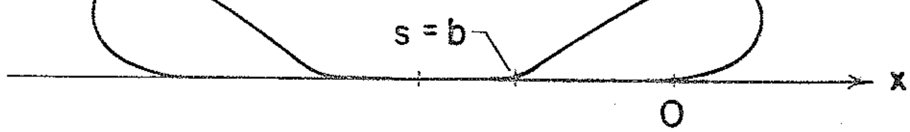



CASE 3 (LINE-POINT CONTACT)




CASE 4 (LINE-LINE CONTACT)