Hanging an Elastic Ring

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Summary: A thin flexible elastic circular ring is hung at one point. The ring deforms due to its own weight. The problem depends on a non-dimensional parameter B representing the relative importance of density and length to rigidity. The heavy elastica equations are solved by perturbation for small B, by a quasi-Newton method for intermediate B, and by a homotopy method for large B. The approximate results show good agreement with numerical integration for B < 20.

Notation

A	FL^2/EI
B	$\rho L^3/EI$
C_i	quasi-Newton Jacobian approximation
EI	flexural rigidity
F	lateral force at $s = 0$
$\underset{\sim}{F}$	vector function
h	height of ring

L	1/2 perimeter of ring
M	moment
N	quasi-Newton update matrix
p_i	search direction for quasi-Newton method
s	arc length
s^*	arc length at $\theta = \pi/2$
t_i	quasi-Newton step size
w	width of ring
$\underset{\sim}{v}, \underset{\sim}{w}$	two-dimensional vectors
x	cartesian coordinate
y	cartesian coordinate
λ	homotopy variable
$\underset{\sim}{\rho}$	homotopy map
α	A/ε
ε	small positive number
ρ	weight-per length
θ	local angle of inclination
$'$	dimensional quantity

Formulation

Suppose a thin, circular elastic ring is hung by a point as shown in Figure 1. Due to its own weight, the ring deforms into a noncircular shape. The present paper studies this deformation as a function of the length, the flexural rigidity, and the density of the ring.

If the thickness of the ring is small compared to its perimeter, the equations of heavy elastica can be used.[1,2]

$$EI\frac{d^2\theta}{ds'^2} = -F\sin\theta - \rho s'\cos\theta \tag{1}$$

$$\frac{dx'}{ds'} = \cos\theta\ , \quad \frac{dy'}{ds'} = \sin\theta \tag{2}$$

The boundary conditions are

$$s' = 0\ , \quad \theta = x' = y' = 0 \tag{3}$$

$$s' = L\ , \quad \theta = \pi\ , \quad x' = 0 \tag{4}$$

We normalize all lengths by L

$$s = s'/L\ , \quad x = x'/L\ , \quad y = y'/L \tag{5}$$

The governing equations become

$$\frac{d^2\theta}{ds^2} = -A\sin\theta - Bs\cos\theta \tag{6}$$

$$\frac{dx}{ds} = \cos\theta \qquad \frac{dy}{ds} = \sin\theta \tag{7}$$

where A,B are nondimensional constants defined by

$$A \equiv \frac{FL^2}{EI} \qquad B \equiv \frac{\rho L^3}{EI} \tag{8}$$

The boundary conditions are

$$s = 0 \qquad \theta = x = y = 0 \tag{9}$$

$$s = 1 \qquad \theta = \pi\ , \quad x = 0 \tag{10}$$

Given B, there are five boundary conditions and five unknowns:

θ , $\frac{d\theta}{ds}$, x , y and A.

Eq (6-10) are impossible to solve in closed form.

Approximate Solutions for Small B

The important parameter B measures the relative importance of density and length to rigidity. $B^{1/3}$ represents the ratio of L to the "bending length" $(EI/\rho)^{1/3}$. Elastic rings with the same value of B have the same configuration. Small B means relatively stiff rings with almost circular shape. Since gravity effects are small, we expect A to be small too. Let

$$B \equiv \varepsilon << 1 \quad , \quad A \equiv \alpha\varepsilon \tag{11}$$

where α is of order unity. Then we expand

$$\theta = \theta_o(s) + \varepsilon\theta_1(s) + \ldots \tag{12}$$

$$x = x_o(s) + \varepsilon x_1(s) + \ldots \tag{13}$$

$$y = y_o(s) + \varepsilon y_1(s) + \ldots \tag{14}$$

$$\alpha = \alpha_o + \varepsilon\alpha_1 + \ldots \tag{15}$$

When Eqs (12-15) are substituted into Eq (6-10), the zeroth order equations are

$$\frac{d^2\theta_o}{ds^2} = 0 \quad , \quad \frac{dx_o}{ds} = \cos\theta_o \quad , \quad \frac{dy_o}{ds} = \sin\theta_o \tag{16}$$

$$\theta_o(0) = x_o(0) = y_o(0) = 0 \quad , \quad \theta_o(1) = \pi \quad , \quad x_o(1) = 0 \tag{17}$$

The solution is

$$\theta_o = \pi s \qquad x_o = \frac{\sin \pi s}{\pi} \qquad y_o = \frac{1 - \cos \pi s}{\pi} \tag{18}$$

The first order equations are

$$\frac{d^2\theta_1}{ds^2} = \alpha_o \sin\theta_o - s\cos\theta_o \quad , \quad \frac{dx_1}{ds} = -\theta_1 \sin\theta_o \quad ,$$

$$\frac{dy_1}{ds} = \theta_1 \cos\theta_o \tag{19}$$

$$\theta_1(0) = x_1(0) = y_1(0) = 0 \qquad \theta_1(1) = x_1(1) = 0 \tag{20}$$

The solution is

$$\theta_1 = \frac{s}{\pi^2} - \frac{3\sin \pi s}{2\pi^3} + \frac{s\cos \pi s}{\pi^2} \tag{21}$$

$$x_1 = \frac{3\ s}{4\pi^3} - \frac{\sin\ 2\ \pi s}{2\pi^4} + \frac{s\ \cos\ 2\ \pi s}{4\pi^3} + \frac{s\ \cos\ \pi s}{\pi^3} - \frac{\sin\ \pi s}{\pi^4} \tag{22}$$

$$y_1 = \frac{s^2}{4\pi^2} - \frac{3}{2\pi^4} + \frac{s\ \sin\ 2\ \pi s}{4\pi^3} + \frac{\cos\ 2\ \pi s}{2\pi^4} + \frac{s\ \sin\ \pi s}{\pi^3} + \frac{\cos\ \pi s}{\pi^4} \tag{23}$$

$$\alpha_0 = -\frac{1}{2\pi} \tag{24}$$

$d\theta/ds$ represents the moment normalized by EI/L^2.

$$\frac{d\theta}{ds} = \pi + \varepsilon\left(\frac{1}{\pi^2} - \frac{\cos\ \pi s}{2\pi^2} - \frac{s\ \sin\ \pi s}{\pi}\right) + O(\varepsilon^2) \tag{25}$$

The maximum moments occuring at the extremum points s=0 and s=1 are

$$\frac{d\theta}{ds}(0) = \pi + \varepsilon\left(\frac{1}{2\pi^2}\right) + O(\varepsilon^2) \tag{26}$$

$$\frac{d\theta}{ds}(1) = \pi + \varepsilon\left(\frac{3}{2\pi^2}\right) + O(\varepsilon^2) \tag{27}$$

The maximum height of the ring is

$$y(1) = \frac{2}{\pi} + \varepsilon\left(\frac{1}{4\pi^2} - \frac{2}{\pi^4}\right) + O(\varepsilon^2) \tag{28}$$

The maximum width is at $\theta = \pi/2$. From Eqs (12,18,21) we find the corresponding arc length is

$$s^* = \frac{1}{2} + \varepsilon\left(\frac{3}{2\pi^4} - \frac{1}{2\pi^3}\right) + O(\varepsilon^2) \tag{29}$$

Substituting Eq (29) into Eqs (13,18,22), we obtain the maximum width

$$2x(s^*) = \frac{2}{\pi} - \varepsilon\left(\frac{2}{\pi^4} - \frac{1}{2\pi^3}\right) + O(\varepsilon^2) \tag{30}$$

The height to width ratio is thus

$$\frac{h}{w} = 1 + \varepsilon\left(\frac{1}{8\pi} - \frac{1}{4\pi^2}\right) + O(\varepsilon^2) \tag{31}$$

Numerical Solutions

Let

$$\underset{\sim}{v} = [A\,,\ \frac{d\theta}{ds}(0)]$$

and $x(s;\underset{\sim}{v})$, $y(s;\underset{\sim}{v})$, $\theta(s;\underset{\sim}{v})$ be the solution (which depends on $\underset{\sim}{v}$) to the initial value problem Eqs (6-7), (9). Then it is clear that the problem given by Eqs (6-10) is equivalent to the nonlinear system of equations

$$\underset{\sim}{F}(\underset{\sim}{v}) = \begin{pmatrix} x(1;\underset{\sim}{v}) \\ \theta(1;\underset{\sim}{v}) - \pi \end{pmatrix} = \underset{\sim}{0} \qquad (32)$$

The standard approach of solving Eq (32) by Newton's method has two serious shortcomings. Newton's method has a very small domain of convergence for this problem, and it frequently either diverges or converges to a mathematically correct but physically meaningless solution which exhibits cross-overs of the ring. For example, for $B > 230$, changing B by one causes Newton's method to converge to the wrong solution. The second difficulty is that Newton's method requires the Jacobian matrix $D\underset{\sim}{F}(\underset{\sim}{v})$ of $\underset{\sim}{F}(\underset{\sim}{v})$, which is very expensive to compute. Whether $D\underset{\sim}{F}(\underset{\sim}{v})$ is approximated by finite differences[5] or calculated explicitly[3], its evaluation is at least four times as expensive as $\underset{\sim}{F}(\underset{\sim}{v})$.

For intermediate sized B ($20 < B < 300$), Eq (32) was solved by a quasi-Newton method[6,7]. This method approximates the Jacobian matrix relatively cheaply, is reliable, robust, and does not require good initial estimates in general. (Specifically, the code HYBRJ from the MINPACK package developed at Argonne National Laboratory was used.)

In general, a quasi-Newton algorithm operates as follows:

1. Start with an estimate C_o of the Jacobian matrix and an estimate $\underset{\sim}{v}_o$ of the solution.
2. Compute a search direction $\underset{\sim}{p}_i$ by solving $C_i \underset{\sim}{p}_i = -\underset{\sim}{F}(\underset{\sim}{v}_i)$.
3. Compute the next approximation

   $\underset{\sim}{v}_{i+1} = \underset{\sim}{v}_i + t_i \underset{\sim}{p}_i$,

   where t_i is chosen to minimize $||F(\underset{\sim}{v}_i + t\underset{\sim}{p}_i)||$ in some "trust region"[6,7].
4. Update the Jacobian approximation by

   $C_{i+1} = C_i + N$,

   where N is an easily and efficiently computed combination of rank one matrices, elementary matrices, and C_i. See Dennis and Moré[7] for the precise form of N.

Unfortunately, for B large enough (B > 300) quasi-Newton methods also fail unless the change in B is intolerably small. (e.g., starting from the solution for B = 500, HYBRJ fails for B = 500.1) A new homotopy method was used to solve Eq (32) for large B.

This powerful method is globally convergent and does not require a close initial guess. Versions of the method have been previously applied to fluid mechanics[5], nonlinear complimentarity[8], fixed point[9], and continuum mechanics problems[12].

The homotopy method is applied to the nonlinear system of equations (32). The theoretical justification of the algorithm requires fairly deep differential geometry, although the algorithm itself is deceptively simple. Thorough discussions of both the theory and some applications can be found in references 5,8-12.

Define a homotopy map $\phi_{\tilde{w}} : [0,1) \times E^2 \to E^2$ by

$$\phi_{\tilde{w}}(\lambda,\tilde{v}) = \phi(\tilde{w},\lambda,\tilde{v}) = \lambda \tilde{F}(\tilde{v}) + (1-\lambda)(\tilde{v} - \tilde{w}) \tag{33}$$

The supporting theory[9] says that for almost all $\tilde{w}$ (i.e., all $\tilde{w}$ except possibly those in a set of Lebesgue measure zero), the Jacobian matrix $D\phi_{\tilde{w}}$ of $\phi_{\tilde{w}}$ has full rank on

$$\phi_{\tilde{w}}^{-1}(0) = \{(\lambda,\tilde{v}) | 0 \le \lambda < 1 \ , \ \tilde{v} \in E^2 \ , \ \phi_{\tilde{w}}(\lambda,\tilde{v}) = 0\},$$

the set of zeros of $\phi_{\tilde{w}}$ in $\lambda,\tilde{v}$ space. The full rank condition implies that the zero set of $\phi_{\tilde{w}}$ consists of smooth disjoint curves which cannot just "stop" in the interior of $(0,1) \times E^2$ (for elaboration see the figures in Watson[9]). The hope is that there is a zero curve γ of $\phi_{\tilde{w}}$ reaching from a trivial known solution (at $\lambda = 0$) to the desired solution (at $\lambda = 1$). Such a zero curve exists under fairly general hypotheses[8-11], but they are often difficult to verify for practical problems. Nevertheless the homotopy method works well in practice.

The algorithm is conceptually simple: track the zero curve γ of $\phi_{\tilde{w}}$ emanating from $(0,\tilde{v}_o)$, where $\phi_{\tilde{w}}(0,\tilde{v}_o) = 0$, until a point $(1,\bar{\tilde{v}})$ is reached. Then $\bar{\tilde{v}}$ is the solution to (32). This algorithm differs significantly from standard continuation in that λ need not increase along γ, and there are never any "singular points" along γ. The power of the algorithm derives from this ability of λ to both increase and decrease along γ, with turning points posing no special difficulty. γ is the trajectory of the initial value problem

$$\frac{d}{ds}\underset{\sim}{\phi}_{\underset{\sim}{w}}(\lambda(s),\underset{\sim}{y}(s)) = 0, \tag{34}$$

$$\left(\frac{d\lambda}{ds}\right)^2 + \left\|\frac{d\underset{\sim}{y}}{ds}\right\|^2 = 1, \tag{35}$$

$$\lambda(0) = 0 \;,\; \underset{\sim}{y}(0) = \underset{\sim}{y}_0, \tag{36}$$

where s is arc length. Equation (34) is

$$D\underset{\sim}{\phi}_{\underset{\sim}{w}}(\lambda(s),\underset{\sim}{y}(s))\begin{pmatrix}\frac{d\lambda}{ds}\\ \frac{d\underset{\sim}{y}}{ds}\end{pmatrix} = 0,$$

where $D\underset{\sim}{\phi}_{\underset{\sim}{w}}$ is the 2 x 3 Jacobian matrix of $\underset{\sim}{\phi}_{\underset{\sim}{w}}$. $D\underset{\sim}{\phi}_{\underset{\sim}{w}}$ has full rank on the zero curve γ given parametrically by $\lambda(s)$, $\underset{\sim}{y}(s)$. Thus the derivative $(d\lambda/ds, d\underset{\sim}{y}/ds)$ is calculated by finding the kernel of $D\underset{\sim}{\phi}_{\underset{\sim}{w}}$, and then using (35) and the continuity of the derivative[3,9,10]. A sophisticated variable step, variable order ordinary differential equation solver is used to solve Eqs (34-36), where the derivatives required by the ODE solver are calculated as just described. Such an ODE solver is very efficient, and considerable computational experience indicates that this approach is superior to schemes using Newton's method and/or simpler ODE techniques to track γ[9-11]. The computer code used was subroutine FIXPT from Watson and Fenner[10].

Results and Discussion

The maximum moment is shown in Fig. 2. The largest moment or curvature occurs at the top at $s = 1$. The height and width are shown in Fig. 3. We see that our solutions for small B, theoretically good for $B \ll 1$, can be extended to $B < 20$ without appreciable error. This is due to the fact that the corrections θ, x, y, are numerically small, thus extending the actual range of validity of the perturbations. Fig. 4 shows the integrated shapes for various values of B. It is seen that as B increases (an increase in length, density or a decrease in rigidity) the shape changes from circular to an oblong pear shape.

Fig. 5 is important since it can be used to inversely determine the flexural rigidity of flexible rings. The procedure is to hang the ring and measure the height to width ratio. Since ρ and L are easily measured, EI can be obtained from B. This method was suggested by Pierce[4] in testing textiles. The theoretical analysis is now presented in the present paper.

Figure Captions

Fig. 1 The coordinate system

Fig. 2 Maximum moment at s=0 and s=1

Fig. 3 The height h and width w of the ring

Fig. 4 The shape of the heavy ring for various values of B, a:B=1, b:B=20, c:B=100, d:B=600 .

Fig. 5 Height to width ratio as a function of B.

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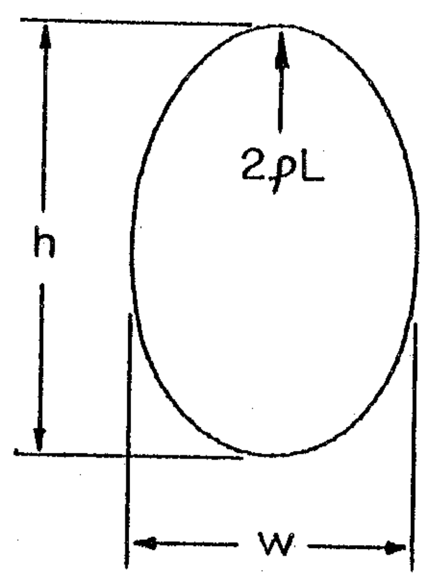

h
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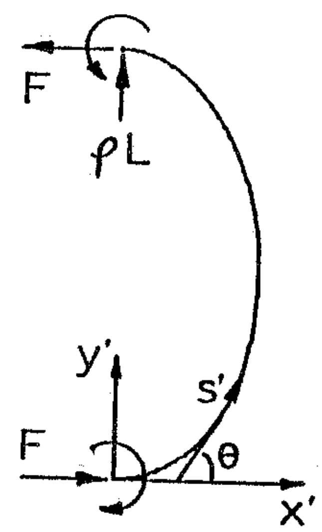
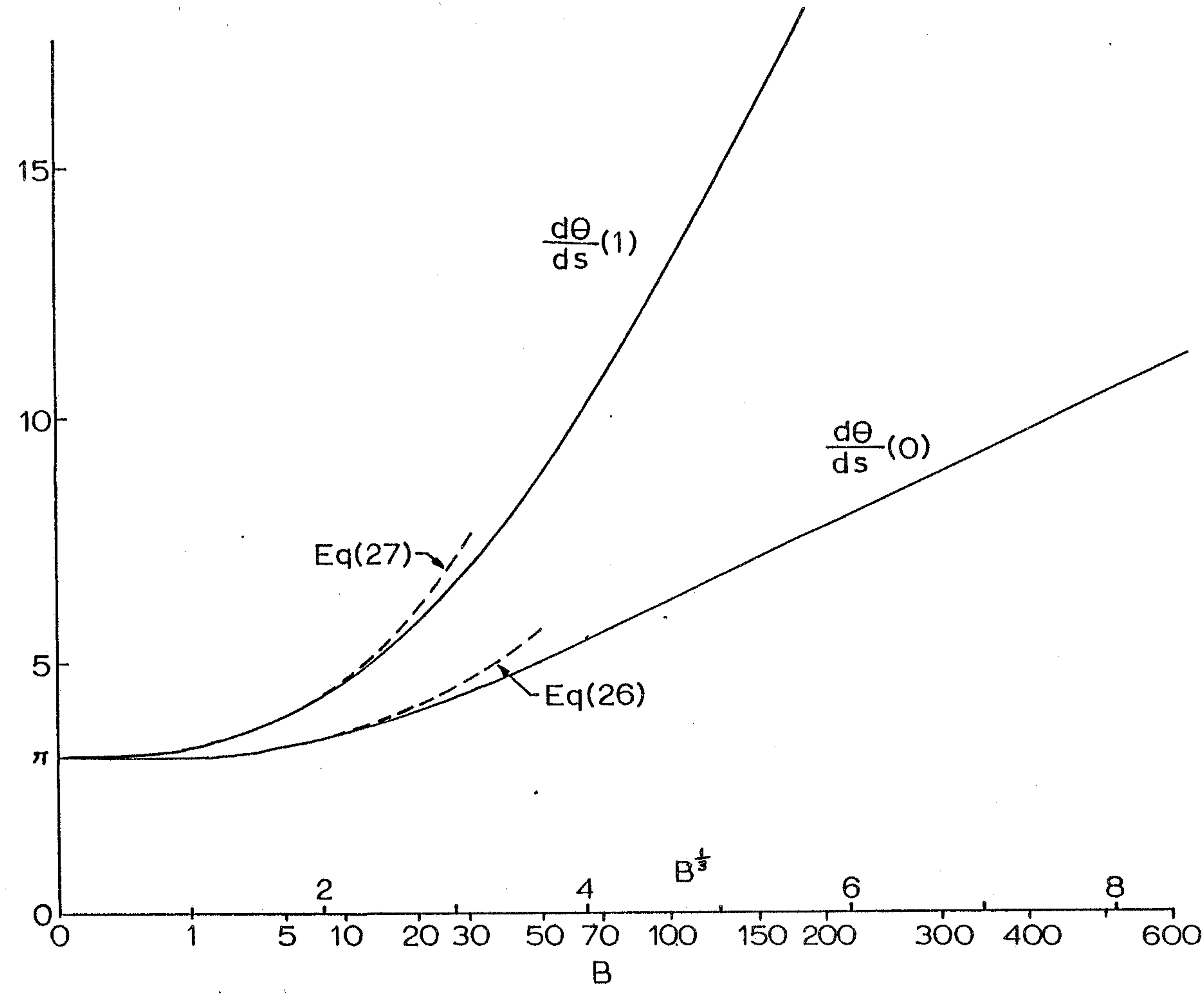
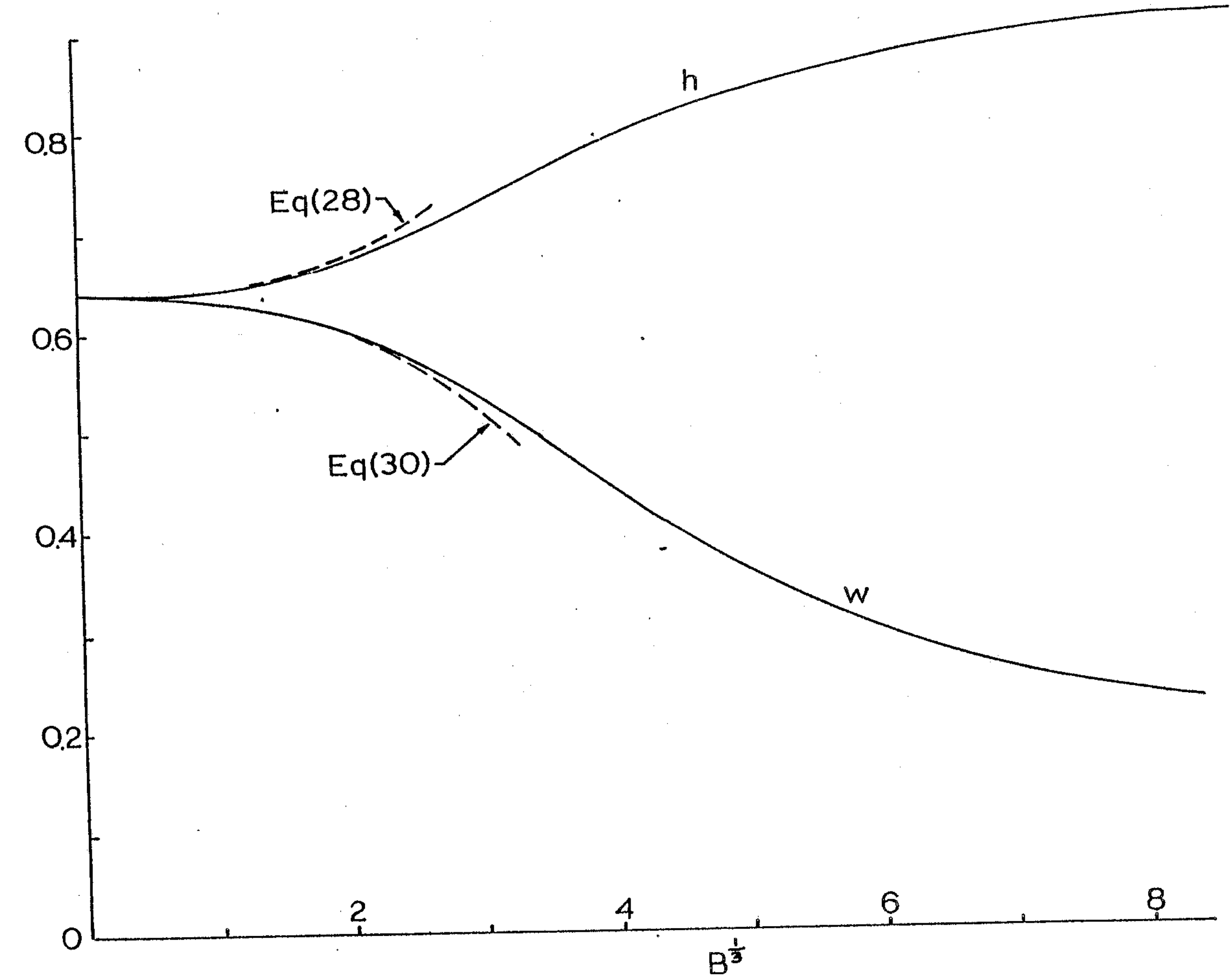
w



F
ρL
y'
s'
F
θ
x'



15
10
5
π
0
$\frac{d\theta}{ds}(1)$
$\frac{d\theta}{ds}(0)$
Eq(27)
Eq(26)
$B^{\frac{1}{3}}$
2
4
6
8
0
1
5
10
20
30
50
70
100
150
200
300
400
600
B


h
w
Eq(28)
Eq(30)
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0.4
0.2
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2
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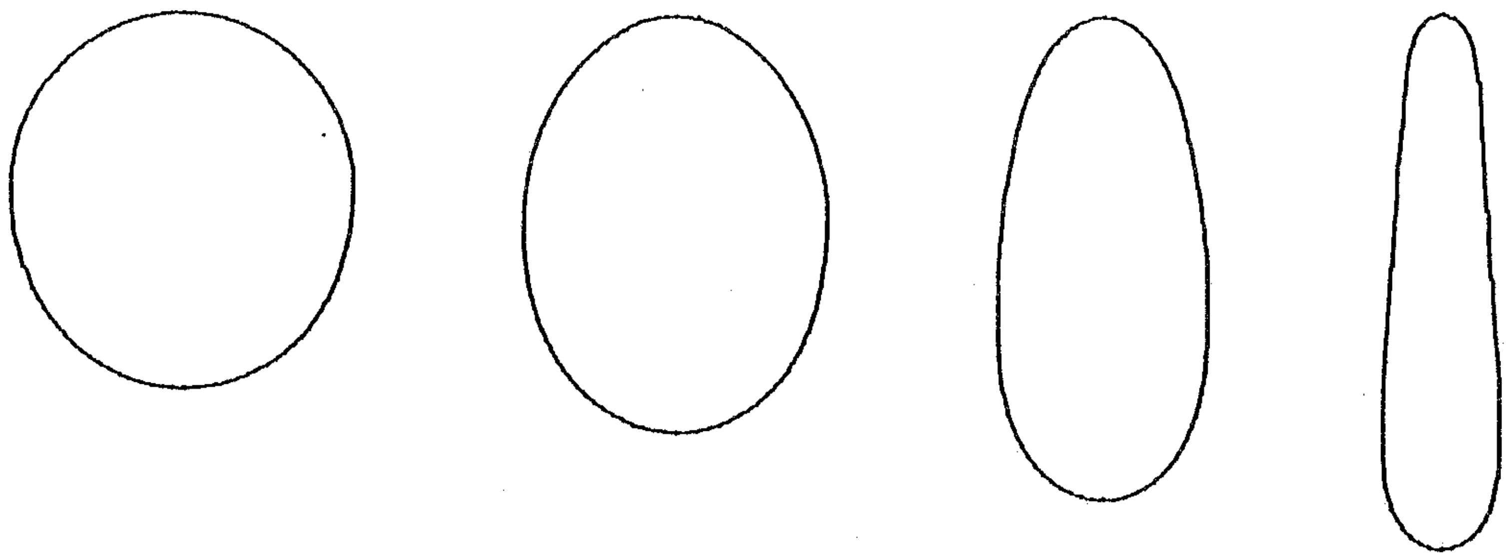
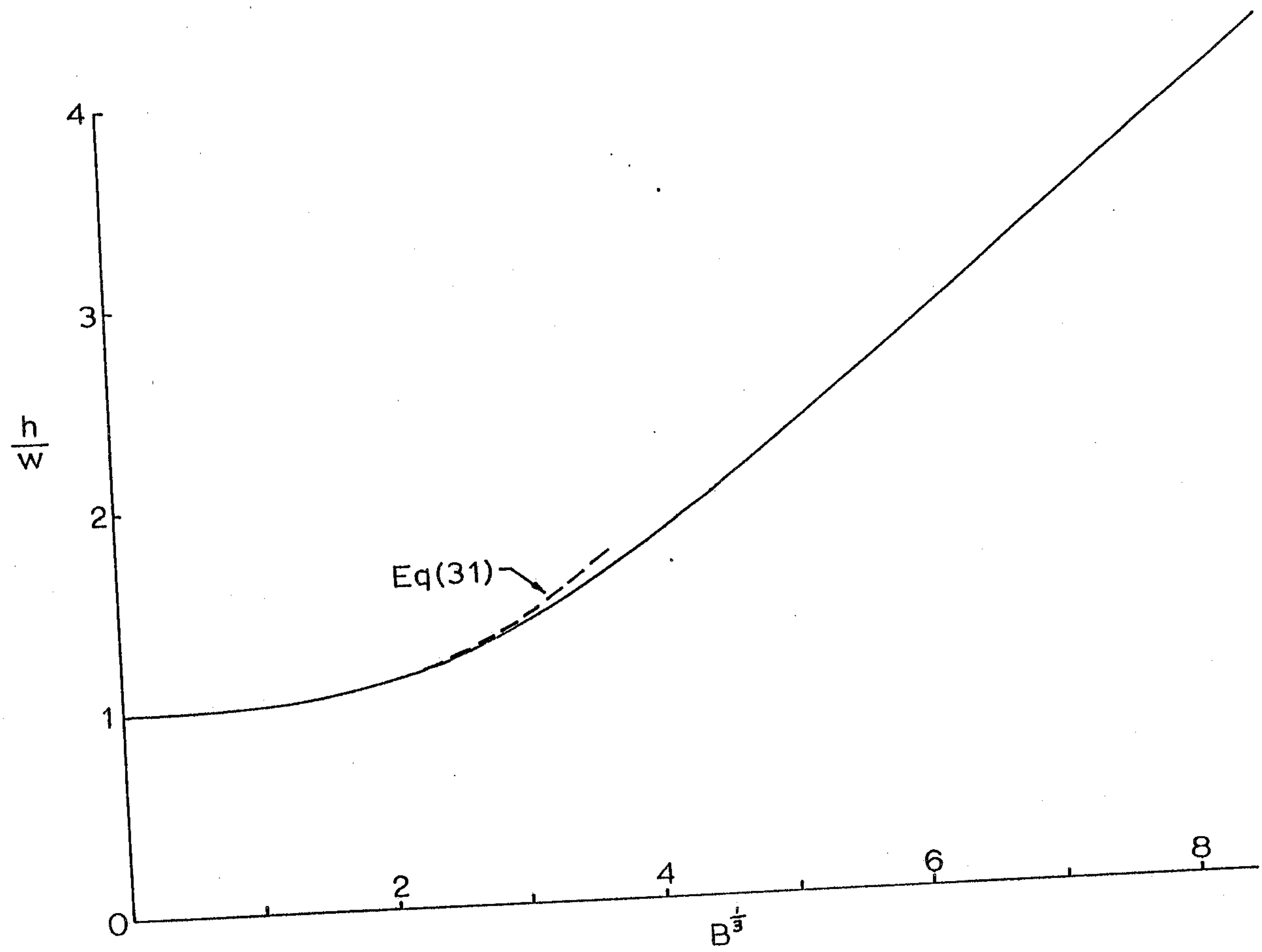
$B^{\frac{1}{3}}$



a b c d



4
3
2
1
0
h/w
Eq(31)
2
4
6
8
B^(1/3)