CS 77001-R

Concerning Transforms For Three-Valued Systems

Jon C. Muzio

Virginia Polytechnic Institute and State University
and
University of Manitoba

February 1977

This report is an initial working paper (#277/1) in connection with a joint project initiated in January of 1977 by S. L. Hurst (University of Bath, England) and J. C. Muzio.

Concerning Transforms for Three-valued Systems

1. Introduction

In this interim report none of the background material is given, nor are any references included. This is in the nature of a working paper and consequently the treatment is not intended to be either complete or rigorous. Indeed it is not clear to the author at this time whether the approach described below is of any value in the design of ternary switching circuits and the decomposition of ternary functions.

The attempt here is to generalize spectral transforms of the Rademacher-Walsh type to a three-valued system. Inevitably something will be lost in the particular generalization pursued and in this case we loose the strict orthogonality of the transform. However the transform is nonsingular and the inverse is trivially deduced from it so this is probably not a serious loss.

We will follow the 2-valued definition of the the transform using the Hadamard ordering viz:

$$\Delta_o = [1]$$

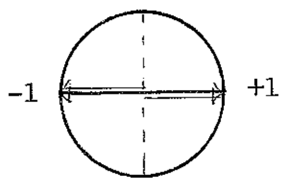
$$\Delta_n = \begin{bmatrix} \Delta_{n-1} & \Delta_{n-1} \\ \Delta_{n-1} & \Delta_{n-1} \end{bmatrix} \qquad \text{for each } n = 1, 2, \ldots$$

The ternary functions we are considering will be defined on $\{0, 1, 2\}$. A function of n variables will be defined by a combination table containing 3^n rows. The right hand column of this table will be called the specification vector for the function and denoted $\underset{\sim}{F}$ for a function $f(x_1,\ldots,x_n)$. The ordering of the rows of the table are on the basis that x_1 change first and

x_n last.

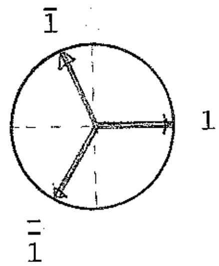
2. The System

In the two-valued environment a function defined on $\{0, 1\}$ is transformed by a matrix defined on $\{+1, -1\}$ and the function is coded into the same set. These values +1, -1 can be considered opposite ends of the main diameter of the unit circle,



or expressed in polar coordinates (r, θ) as $(1, 0)$, $(1, \pi)$.

A reasonable generalization to three values appears to be to use three equidistant points on the unit circle $(1,0)$, $(1, 2\pi/3)$, $(1, 4\pi/3)$ which we will call 1, $\bar{1}$, and $\bar{\bar{1}}$ respectively.



Using the normal techniques for addition, multiplication by a scalar, and multiplication, namely that addition is normal vector addition and

$$k(r,\theta) = (kr,\theta) \quad ,$$

$$(r_1, \theta_1) \cdot (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2)$$

we have for our $\{1, \bar{1}, \bar{\bar{1}}\}$ system that $1.x = x.1 = x$ for all x

$$\bar{1}.\bar{1} = \bar{\bar{1}}$$

$$\bar{\bar{1}}.\bar{\bar{1}} = \bar{1}$$

$$\bar{1}.\bar{\bar{1}} = \bar{\bar{1}}.\bar{1} = 1$$

and $\quad 1 + \bar{1} + \bar{\bar{1}} = 0$

so that, for example, $1 + \bar{1} = -\bar{\bar{1}}$ etc.

$(-(r,\theta) = (r,\theta + \pi))$.

This leads to some slightly curious looking arithmetic, for example

$$\bar{\bar{5}} + \bar{3} + 2 = \bar{\bar{3}} + \bar{1} \qquad \text{since } \bar{\bar{2}} + \bar{2} + 2 = 0$$

$$= \bar{\bar{2}} - 1 \qquad \text{since } \bar{\bar{1}} + \bar{1} = -1$$

We are going to code our $\{0, 1, 2\}$ system into the $\{1, \bar{1}, \bar{\bar{1}}\}$ system in a manner analagous to the way in which the $\{0, 1\}$ system was coded into $\{\bar{1}, 1\}$ system for two-valued functions.

3. The Transform

The definition is an obvious generalization of that for the two-valued case.

$$\Delta_o = [1]$$

$$\Delta_n = \begin{bmatrix} \Delta_{n-1} & \Delta_{n-1} & \Delta_{n-1} \\ \Delta_{n-1} & \bar{\Delta}_{n-1} & \bar{\bar{\Delta}}_{n-1} \\ \Delta_{n-1} & \bar{\bar{\Delta}}_{n01} & \bar{\Delta}_{n-1} \end{bmatrix}$$

for each $n = 1, 2, \ldots.$

where Δ_{n-1} means multiplication of every element in Δ_{n-1} by $\bar{1}$ and similarly for $\bar{\bar{\Delta}}_{n-1}$ (in the 2-valued case of course $-\Delta_{n-1}$ meant the multiplication of every element in Δ_{n-1} by -1).

The first 3 transforms are listed below.

We can make a number of observations concerning the transforms.

(i) Δ_n is a 3^n x 3^n matrix

(ii) Δ_n is symmetric

(iii) Δ_n is not orthogonal. However if we denote by $\Delta_n{}^*$ the result of interchanging $\bar{1}$ and $\bar{\bar{1}}$ in Δ_n then it is clear that $\Delta_n{}^{-1} = \Delta_n{}^*$ except for a constant scaling factor of $1/3^n$. Hence

$$\Delta_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \bar{1} & \bar{\bar{1}} \\ 1 & \bar{\bar{1}} & \bar{1} \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} \\ 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} \\ 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\ 1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} \\ 1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 \\ 1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 \\ 1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} \\
1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} \\
1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} \\
1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & 1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & 1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 \\
1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} & 1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} & 1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & 1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & 1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 \\
1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} & 1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} & 1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\
1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} \\
1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 \\
1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 \\
1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & 1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & 1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} \\
1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & 1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} & 1 & 1 & 1 \\
1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & 1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & 1 & \bar{1} & \bar{\bar{1}} \\
1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & 1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} & 1 & \bar{\bar{1}} & \bar{1} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\
1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 \\
1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} \\
1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & 1 & 1 \\
1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} & 1 & \bar{1} & \bar{\bar{1}} \\
1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & 1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & 1 & \bar{\bar{1}} & \bar{1} \\
1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & 1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 & 1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} \\
1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}} & 1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & 1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1
\end{bmatrix}$$

if $R = \Delta_n \underset{\sim}{F}$ then $\underset{\sim}{F} = \frac{1}{3^n} \Delta_n^{*} \underset{\sim}{R}$.

If we identify rows of the transform with variables in the same way as the Rademacher functions then the other rows can be deduced in the same way as the Walsh functions are in the two-valued case. For example for Δ_2 we have

$$
\begin{array}{ll}
 & R_0 \\
 & R_1 \\
s_1 & \\
 & R_2 \\
s_2 & \\
s_3 & \\
s_4 & \\
s_5 & \\
s_6 &
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} \\
1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} \\
1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} \\
1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 \\
1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 \\
1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}}
\end{bmatrix}
$$

and $s_1 = R_1 \cdot R_1$ (in this case of course $R_1 \cdot R_1$ does not retrieve R_0 again as it does in the two-valued case --- now we need $R_1 \cdot R_1 \cdot R_1$ to retrieve R_0).

By $R_1 \cdot R_1$ we mean of course the products of the corresponding entries in each row.

$s_2 = R_2 \cdot R_1$

$s_3 = R_2 \cdot R_1{}^2$

$s_4 = R_2 \cdot R_2$

$s_5 = R_2 \cdot R_2 \cdot R_1$

$s_6 = R_2 \cdot R_2 \cdot R_1 \cdot R_1$

We could label the rows (and the resulting coefficients) as in the 2-valued case giving R_0 , R_1 , R_{11}, R_2, R_{21}, R_{211}, R_{22}, R_{221}, and R_{2211}, respectively for the nine rows. The question as to whether the coefficients will give any correlation with three-valued switching functions will be briefly considered in the next section.

The 27 rows for Δ_3 will be labelled R_0, R_1, R_{11}, R_2, R_{12}, R_{112}, R_{22}, R_{122}, R_{1122}, R_3, R_{13}, R_{113}, R_{23}, R_{123}, R_{1123}, R_{2231}, R_{1223}, R_{11223}, R_{33}, R_{133}, R_{1133}, R_{233}, R_{1233}, R_{11233}, R_{2233}, R_{12233}, R_{112233} .

(note: we are using $R_{(\,)}$ to represent three different things without bothering to change the notation.

(i) the row of the transform

(ii) the spectral coefficient resulting form $\Delta_n \underset{\sim}{F}$

(iii) the function which is correlated with $\underset{\sim}{F}$ by the transform.

The context will make clear which of these applies at any particular time.)

4. The Interpretation for Ternary Functions

Let us consider the interpretation for ternary switching functions and whether $\Delta_n \underset{\sim}{F}$ is giving us any useful information in terms of the decomposition of ternary functions.

Initially we consider the functions which the transform is comparing a given function with.

Let us rewrite the 2nd order transform putting an identification of the variables across the top.

$$\begin{array}{cc}
\begin{array}{c} x_1 \\ x_2 \end{array} & \begin{array}{ccccccccc} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{array} \\
\begin{array}{c} R_0 \\ R_1 \\ R_{11} \\ R_2 \\ R_{12} \\ R_{112} \\ R_{22} \\ R_{122} \\ R_{1122} \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} \\
1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} \\
1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} \\
1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 \\
1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 \\
1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}}
\end{bmatrix}
\end{array}$$

Since R_1 must provide exact correlation with x_1 and R_2 with x_2 we must identify $1 \leftrightarrow 0$, $\bar{1} \leftrightarrow 1$, $\bar{\bar{1}} \leftrightarrow 2$ in this transform.

The nine functions in the transform are listed below:

	R_0	0	1	2	x_2
	0	0	0	0	
x_1	1	0	0	0	
	2	0	0	0	

	R_1	0	1	2	x_2
	0	0	0	0	
x_1	1	1	1	1	
	2	2	2	2	

	R_{11}	0	1	2	x_2
	0	0	0	0	
x_1	1	2	2	2	
	2	1	1	1	

	R_2	0	1	2	x_2
	0	0	1	2	
x_1	1	0	1	2	
	2	0	1	2	

	R_{12}	0	1	2	x_2
	0	0	1	2	
x_1	1	1	2	0	
	2	2	0	1	

	R_{112}	0	1	2	x_2
	0	0	1	2	
x_1	1	2	0	1	
	2	1	2	0	

	R_{22}	0	1	2	x_2
	0	0	2	1	
x_1	1	0	2	1	
	2	0	2	1	

	R_{122}	0	1	2	x_2
	0	0	2	1	
x_1	1	1	0	2	
	2	2	1	0	

	R_{1122}	0	1	2	x_2
	0	0	2	1	
x_1	1	2	1	0	
	2	1	0	2	

Of these R_0 represents a correlation with a constant 0; R_1 , R_{11} , R_2 , and R_{22} are all functions of a single variable; R_{12} , R_{112} , R_{122} , and R_{1122} are functions of two variables and do represent kinds of generalizations of XOR functions in that for example, R_{12} represents $x_1 + x_2$ (mod 3).

The three-variable functions considered in the third order transformation are given in the table below:

x_1	x_2	x_3	R_0	R_1	R_{11}	R_2	R_{12}	R_{112}	R_{22}	R 122	R 1122	R 3	R 13	R 113	R 23	R 123	R 1123	R 223	R 1223	R 11223	R 33	R 133	R 1133	R 233	R 1233	R 11233	R 2233	R 12233	R 112233
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
2	0	0	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1	0	2	1
0	1	0	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
1	1	0	0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1	0	1	2	1	2	0	2	0	1
2	1	0	0	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0	0	2	1	1	0	2	2	1	0
0	2	0	0	0	0	2	2	2	1	1	1	0	0	0	2	2	2	1	1	1	0	0	0	2	2	2	1	1	1
1	2	0	0	1	2	2	0	1	1	2	0	0	1	2	2	0	1	1	2	0	0	1	2	2	0	1	1	2	0
2	2	0	0	2	1	2	1	0	1	0	2	0	2	1	2	1	0	1	0	2	0	2	1	2	1	0	1	0	2
0	0	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
1	0	1	0	1	2	0	1	2	0	1	2	1	2	0	1	2	0	1	2	0	2	0	1	2	0	1	2	0	1
2	0	1	0	2	1	0	2	1	0	2	1	1	0	2	1	0	2	1	0	2	2	1	0	2	1	0	2	1	0
0	1	1	0	0	0	1	1	1	2	2	2	1	1	1	2	2	2	0	0	0	2	2	2	0	0	0	1	1	1
1	1	1	0	1	2	1	2	0	2	0	1	1	2	0	2	0	1	0	1	2	2	0	1	0	1	2	1	2	0
2	1	1	0	2	1	1	0	2	2	1	0	1	0	2	2	1	0	0	2	1	2	1	0	0	2	1	1	0	2
0	2	1	0	0	0	2	2	2	1	1	1	1	1	1	0	0	0	2	2	2	2	2	2	1	1	1	0	0	0
1	2	1	0	1	2	2	0	1	1	2	0	1	2	0	0	1	2	2	0	1	2	0	1	1	2	0	0	1	2
2	2	1	0	2	1	2	1	0	1	0	2	1	0	2	0	2	1	2	1	0	2	1	0	1	0	2	0	2	1
0	0	2	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
1	0	2	0	1	2	0	1	2	0	1	2	2	0	1	2	0	1	2	0	1	1	2	0	1	2	0	1	2	0
2	0	2	0	2	1	0	2	1	0	2	1	2	1	0	2	1	0	2	1	0	1	0	2	1	0	2	1	0	2
0	1	2	0	0	0	1	1	1	2	2	2	2	2	2	0	0	0	1	1	1	1	1	1	2	2	2	0	0	0
1	1	2	0	1	2	1	2	0	2	0	1	2	0	1	0	1	2	1	2	0	1	2	0	2	0	1	0	1	2
2	1	2	0	2	1	1	0	2	2	1	0	2	1	0	0	2	1	1	0	2	1	0	2	2	1	0	0	2	1
0	2	2	0	0	0	2	2	2	1	1	1	2	2	2	1	1	1	0	0	0	1	1	1	0	0	0	2	2	2
1	2	2	0	1	2	2	0	1	1	2	0	2	0	1	1	2	0	0	1	2	1	2	0	0	1	2	2	0	1
2	2	2	0	2	1	2	1	0	1	0	2	2	1	0	1	0	2	0	2	1	1	0	2	0	2	1	2	1	0

If we write $a \oplus b$ for $a + b \pmod 3$ then all the functions in the transform are $\oplus$ functions. Indeed they are all the possible $\oplus$ functions from n variables (for Δ_n).

For example

R_{11} is $x_1 \oplus x_1$ (which of course is not always 0 - $x_1 \oplus x_1 \oplus x_1$ would be)

R_{12} is $x_1 \oplus x_2$

R_{11233} is $x_1 \oplus x_1 \oplus x_2 \oplus x_3 \oplus x_3$ or maybe we should write such functions as $2x_1 \oplus x_2 \oplus 2x_3$.

At least we have some interpretation for the function in the transform. Let us now turn our attention to function $\underset{\sim}{F}$.

Initially we have to decide how to code $\underset{\sim}{F}$. It cannot be coded the same way that we intrepreted the transform $0 \leftrightarrow 1$, $1 \leftrightarrow \bar{1}$, $2 \leftrightarrow \bar{\bar{1}}$ for essentially the reason that the transform is not orthogonal, viz. if we use that coding we shall not get the maximum value from e.g. R_1 if the function is equal to x_1. The representation for x_1 must be such that when the transform operates on it we get a maximum value for R_1 and 0 for all the rest of the coefficients. There appear to be several possibilities e.g.

(i) $0 \leftrightarrow \bar{\bar{1}}$, $1 \leftrightarrow \bar{1}$, $2 \leftrightarrow 1$

(ii) $0 \leftrightarrow 1$, $1 \leftrightarrow \bar{\bar{1}}$, $2 \leftrightarrow \bar{1}$

There are certainly other possibilities. It is not clear which of these is likely to be most useful or indeed the exact relation between the spectra under the different coding schemes.

We will use (i) above for the examples below. All the examples will be two-variable.

$$\Delta_2 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} & 1 & \bar{1} & \bar{\bar{1}} \\
1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} & 1 & \bar{\bar{1}} & \bar{1} \\
1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & 1 & \bar{\bar{1}} & 1 & \bar{1} \\
1 & \bar{\bar{1}} & \bar{1} & \bar{1} & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & 1 \\
1 & 1 & 1 & \bar{\bar{1}} & \bar{\bar{1}} & \bar{\bar{1}} & \bar{1} & \bar{1} & \bar{1} \\
1 & \bar{1} & \bar{\bar{1}} & \bar{\bar{1}} & 1 & \bar{1} & \bar{1} & \bar{\bar{1}} & 1 \\
1 & \bar{\bar{1}} & \bar{1} & \bar{\bar{1}} & \bar{1} & 1 & \bar{1} & 1 & \bar{\bar{1}}
\end{bmatrix}$$

<u>Example 1</u> $f = x_1$

For convenience both $\underset{\sim}{F}$ and $\underset{\sim}{R} = \Delta_2 \underset{\sim}{F}$ will be written as row vectors in the examples.

$$\underset{\sim}{F} = [\bar{\bar{1}} \quad \bar{1} \quad 1 \quad \bar{\bar{1}} \quad \bar{1} \quad 1 \quad \bar{\bar{1}} \quad \bar{1} \quad 1]$$

$$\underset{\sim}{R} = [0 \quad \bar{\bar{9}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

<u>Example 2</u> $f = x_1 \oplus 1$ (cyclic negation)

$$\underset{\sim}{F} = [\bar{1} \quad 1 \quad \bar{\bar{1}} \quad \bar{1} \quad 1 \quad \bar{\bar{1}} \quad \bar{1} \quad 1 \quad \bar{\bar{1}}]$$

$$\underset{\sim}{R} = [0 \quad \bar{9} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

x_1 \ x_2	0	1	2
0	1	1	1
1	2	2	2
2	0	0	0

<u>Example 3</u> $f = \max(x_1, x_2)$

$$\underset{\sim}{F} = [\bar{\bar{1}} \quad \bar{1} \quad 1 \quad \bar{1} \quad \bar{1} \quad 1 \quad 1 \quad 1 \quad 1]$$

$$\underset{\sim}{R} = [4 + \bar{2},\ \bar{1} + \bar{\bar{5}},\ 1 + \bar{2},\ \bar{1} + \bar{\bar{5}},\ 2 + \bar{\bar{4}},\ \bar{1} + \bar{\bar{2}},\ 1 + \bar{2},\ \bar{1} + \bar{\bar{2}},\ 2 + \bar{\bar{1}}]$$

x_1 \ x_2	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

The R_0 term appears to give some idea of the balance of the table related to 0, viz in this case 4 more 2's than 0's and 2 more 1's than 0's (the 2's being coded by 1, the 1's by $\bar{1}$).

It is not clear to the author exactly what useful information may be gleaned from the spectral coefficients, except that those which are in some sense "larger" viz R_1, R_2 and R_{12} which are $\bar{1} + \bar{\bar{5}}$, $\bar{1} + \bar{\bar{5}}$, and $2 + \bar{\bar{4}}$ respectively are much closer to the function than the others. Indeed the tables agree in 6, 6, and 5 places respectively.

<u>Example 4</u> $\quad f = \min(x_1, x_2)$

$$\underset{\sim}{F} = [\bar{\bar{1}} \;\; \bar{\bar{1}} \;\; \bar{\bar{1}} \;\; \bar{\bar{1}} \;\; \bar{1} \;\; \bar{1} \;\; \bar{\bar{1}} \;\; \bar{1} \;\; 1]$$

$$\underset{\sim}{R} = [\bar{2} + \bar{\bar{4}},\ 1 + \bar{\bar{5}},\ 1 + \bar{\bar{2}},\ 1 + \bar{\bar{5}},\ 2 + \bar{4},\ 2 + \bar{1},\ 1 + \bar{\bar{2}},\ 2 + \bar{1},\ \bar{2} + \bar{\bar{1}}]$$

$x_1 \backslash x_2$	0	1	2
0	0	0	0
1	1	1	1
2	0	1	2

Again the coefficients appear to give very useful information as to the relation of f to our test functions. However the exact nature of this relationship is not transparent to me and the intricacies will require considerable study. Whether there is any possibility of spectral translation techniques being applied to such spectra is a question still to be investigated.