TECHNICAL REPORT NO. CS 75006-R

AN EXACT UPDATE FOR HARRIS' TREAD

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April 1975

ABSTRACT

The purpose of this note is to show how Harris' TREAD value can be computed without approximation.

Computing Reviews No. 5-41

Keywords: linear programming, optimization, pricing

In this note we are concerned with a pricing tactic due to Harris (1) which is used in a linear programming system to decrease computational cost. We begin with a description of the basic idea.

At the beginning of a major iteration a π-vector is computed with a BTRAN algorithm such that the reduced cost of activity j is the inner product, (π, A_j), where A_j is the j-th (original) column. An activity is rejected if the sign of its reduced cost does not show a rate of improvement (depending upon whether the nonbasic activity level is at lower bound or upper bound). If it is not rejected, then it competes with other candidates which have also not been rejected. One criteria upon which this competition is based is the magnitude of the reduced cost.

Denote the reduced cost of activity j by 'D_j' (so $D_j = (\pi, A_j)$). Harris noted that D_j depends upon column scale. That is, if $\tilde{A}_j = s_j A_j$, where $s_j > 0$ is a scale factor, then

$$\tilde{D}_j = (\pi, \tilde{A}_j) = s_j (\pi, A_j) = s_j D_j.$$

Therefore, using magnitude of reduced cost as the criterion for competing activities to enter the basis may be misleading. Indeed, experience confirms this.

As an alternative one may consider the dynamic scaling

$$D_j' = D_j / \|\alpha^j\| ,$$

where $\alpha^j = B^{-1} A_j$. Note that D_j' does not depend upon scaling the j-th column since

$$\tilde{\alpha}^j \equiv B^{-1}\tilde{A}_j = s_j B^{-1} A_j = s_j \alpha^j$$

so

$$\|\tilde{\alpha}^j\| = s_j \|\alpha^j\|.$$

A less superficial rationale for using D_j' is as follows. Let t be the change in activity level of nonbasic activity j; let β be the current basic activity levels; and let Z(t) be the basic activity levels satisfying

$$Z(t) = \beta - t\ \alpha^j .$$

Then, the total distance moved by the basic variables is

$$D(t) = \| Z(t) - Z(0) \| = t\|\alpha^j\| .$$

Let $Z_o(t)$ be the objective row activity, and define

$$\text{GRADIENT} \equiv \left. \frac{dZ_o(t)}{dD} \right|_{t=o}$$

That is, the GRADIENT is the initial rate of change in the objective value with respect to change in total distance. Then,

$$\text{GRADIENT} = \left. \frac{dZ_o}{dt} \bigg/ \frac{dD}{dt} \right|_{t=o}$$

$$= D_j / \|\alpha^j\| .$$

Thus, Harris' criterion is measuring a different rate than the reduced cost not only to account for scale dependance but more importantly the geometry associated with the basis reflected by α. She calls $\|\alpha^j\|$ the TREAD and $|D_j|$ the RISE.

Harris' algorithm has some extra complication by maintaining a set of "reference variables" but the essence of her method is to use the GRADIENT. Another variation we have taken liberty to use is the use of an "auxiliary π-vector. Harris keeps the objective row current, so she only BTRANs one vector to approximate TREAD. have assumed it may be overall better to perform multiple BTRAN.

Given we want to use a GRADIENT measure the issue now is how to compute the TREAD without first computing α^j for each candidate since that would be prohibitively costly. Let us work with the square, $T_j = \|\alpha^j\|^2$, to avoid square root computation.

For any basic variable we know its α-vector is a unit vector (i.e., $B^{-1} B_i = e_i$), so $T_i = 1$ for basic activity i. For an all-logical basis we have $\alpha^j = A_j$, so

$T_j = \|A_j\|^2$ initially (if we begin with an all-logical basis). This can be computed at setup time once and for all, and the remaining issue is how to update T_j when the basis changes.

Let α and α' be the α-vectors for the old and new bases, respectively, when A_q enters, replacing the p-th basic variable. (Note j-superscripts suppressed). Further, let $\tilde{\alpha}$ be the α-vector of A_q from the old basis, where $\tilde{\alpha}$ was computed at pivot row selection time.

Then, we have the pivot equations,

$$\alpha_p' = \alpha_p / \tilde{\alpha}_p$$
$$\alpha_i' = \alpha_i - \tilde{\alpha}_i \alpha_p' \qquad \text{for } i \neq p.$$

Defining

$$\theta \equiv \alpha_p / \tilde{\alpha}_p$$

we can write the above in vector form as

$$\alpha' = \alpha - \theta \tilde{\alpha} + \theta e_p ,$$

where e_p is the p-th column of the identity matrix.

Therefore,

$$T_j' = (\alpha', \alpha') = (\alpha, \alpha) + \theta^2 (\tilde{\alpha}, \tilde{\alpha}) + \theta^2 - 2\theta(\alpha, \tilde{\alpha}) + 2\theta\alpha_p - 2\theta^2\tilde{\alpha}_p$$
$$= T_j + \theta^2 (T_q + 1) - 2\theta(\alpha, \tilde{\alpha}).$$

Harris now considers this last equation. We have the saved values, T_j and T_q, for activities j and q, respectively. We do not know θ and $(\alpha, \tilde{\alpha})$. However, at the major iteration we can BTRAN e_p to obtain $\pi' = e_p B^{-1}$, and we note

$$(\pi', A_j) = (e_p B^{-1}, A_j) = (e_p, B^{-1} A_j) = (e_p, \alpha) = \alpha_p .$$

Therefore, by using an auxiliary π-vector, denoted π', we can obtain α_p during pricing. Since $\tilde{\alpha}$ was computed, we can save $\tilde{\alpha}_p$ and thereby compute θ ($= \alpha_p / \tilde{\alpha}_p$) during pricing.

At this point Harris drops the remaining term and uses the approximation update,

$$T_j' \cong T_j + \theta^2 (T_q + 1).$$

If $\theta = 0$ or α is orthogonal to $\tilde{\alpha}$, then her update is exact; otherwise, it is not. Let us examine the dropped term more closely.

Consider the value of $(\alpha, \tilde{\alpha})$ for the activity (j) that just became nonbasic upon the exchange, (p, q). We know $T_j = 1$ since its old α-vector was e_p. Therefore, we have

$$T_j' = 1 + \theta^2 (T_q + 1) - 2\,\theta\,(e_p, \tilde{\alpha}),$$

and

$$2\,\theta\,(e_p, \tilde{\alpha}) = 2\,(1/\tilde{\alpha}_p)\,(\tilde{\alpha}_p) = 2.$$

We see that the dropped term is -2, not an insignificant number!

One immediate modification to Harris' scheme is to use this exact value of T_j for the variable just becoming nonbasic. However, we can go further and show how to obtain $(\alpha, \tilde{\alpha})$ by using another auxiliary π-vector.

We have

$$(\tilde{\alpha}, \alpha) = (\tilde{\alpha}, \alpha' + \theta\tilde{\alpha} - \theta e_p)$$

$$= (\tilde{\alpha}, \alpha') + \theta\, T_q - \theta\, \tilde{\alpha}_p.$$

Now $(\tilde{\alpha}, \alpha')$ is the only remaining unknown, but by using the auxiliary π-vector, $\pi'' = \tilde{\alpha}\,(B')^{-1}$, we have

$$(\pi'', A_j) = (\tilde{\alpha}, \alpha'),$$

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