TECHNICAL REPORT NO. CS 74022-R

GLM Versus Continuous Approximation for Convex Integer Programs

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December 1974

ABSTRACT

GLM is compared to continuous approximation for convex, integer programs. After noting the stronger bound provided by GLM, Lagrangian duality and a gap closing heuristic is used to demonstrate how GLM may provide a better feasible policy as well.

Computing Reviews Number 5.41

Keywords: Optimization, Lagrange Multipliers, Integer Programming, Convex Programming.

We are concerned with a mathematical program of the form:

$$\text{Max } f(x): x \text{ in } S \wedge I \text{ and } g(x) = 0,$$

where f is concave on S, g is convex on S, and I is the set of integer n-vectors. Our concern is with comparing solutions obtained using the Generalized Lagrange Multiplier Method (GLM) and the continuous approximation (CA), where "x in I" is relaxed. For convenience we shall let S be a rectangle with integer extreme points. If we further assume that f and g are affine, then the Nemhauser - Ullman theorem (4) shows the GLM bound equals the CA bound, and a gap prevails if, and only if, the continuous program (which is an ordinary linear program) has no integer optimum.

In a recent note (3) I established a theorem on conjugate bounds which can be specialized to show that the upper bound generated by GLM is stronger than that of the continuous approximation. Falk (1) subsequently showed that if g is concave on S and sum-separable then the two bounds are equal. Thus, in our present model, if g is affine, then the upper bounds by GLM and CA are equal.

However, the affine case notwithstanding, we shall show how GLM may generate better feasible solutions than searching from a continuous approximation. Let S be a rectangle with least element, 0, and let $g(0) < 0$. The Lagrangian duals for GLM and CA, respectively are as follows:

$$G: \quad \text{Min } L^*(y) = \text{Sup}\{f(x) - yg(x): \ x \text{ in } S \wedge I\} \quad \text{for } y \geqq 0$$

$$C: \quad \text{Min } \bar{L}(y) = \text{Sup}\{f(x) - yg(x): \ x \text{ in } S\} \quad \text{for } y \geqq 0.$$

Under our assumptions, both G and C have solutions, say y^* and $\bar{y}$, respectively.

It is noted that if CA yields an integer optimum, then no duality gap prevails for GLM and we may consider $y^* = \bar{y}$. To see this let x be such a policy, and observe:

$$f(x) = \bar{L}(\bar{y}) + \bar{y}\, g(x) = L^*(\bar{y}) + \bar{y}\, g(x).$$

More typically, the set of policies associated with C (ie, where $f(x) = L^*(y) + y\, g(x)$) are never integer. Thus, CA has noninteger policies which satisfy the g-constraints, while GLM produces integer policies which need not satisfy the g-constraints. Of course, in the special case of one constraint at least one GLM policy associated with y* is feasible, but it need not satisfy the complementarity condition: $y^*\, g(x) = 0$.

What advantages does GLM offer in getting a "good" feasible solution? An answer lies in the meaning of a GLM solution. At a minimal multiplier, there are k+1 policies (with $k \leq m$), say $x^0, \ldots, x^k$, which satisfy:

1. $f(x^i) = L^*(y^*) + y^*\, g(x^i)$
2. $\sum_{i=0}^{k} w_i g(x^i) \leq 0$
3. $y^* \sum_{i=0}^{k} w_i\, g(x^i) = 0$

for some weights, $w > 0$ and $\sum_{i=0}^{k} \omega_i = 1$. These can be interpreted as probabilities, and the GLM solution can be interpreted as a mixed strategy.

Now consider the associated average defined by

$$x = \sum_{i=0}^{k} w_i\, x^i$$

Since g is convex, we have

$$g(x) \leq 0,$$

so x is feasible with respect to the g-constraints, but not necessarily integer. If it should turn out that

$$y^*\, g(x) = 0$$

(as when all functions having positive multiplier are affine), then x is a solution to the continuous approximations and y* is a minimal multiplier of its Lagrangian dual (C).

However, while CA only produces the average policy, from which some heuristic method of rounding can be used, GLM produces the base set, $\{x^i\}_0^k$. If, for example, g is monotone increasing, then it is easy to search for a feasible policy from any base point.

A related, but different, heuristic was described in (2) for gap closing. The criteria used for choice of base (from different multipliers used during the course of solving G) was its proximity to the target right-hand-side, which we have canonically taken to be zero. Given a base, coordinates may be decreased or increased to fill slack or remove excess. The Lagrangian aids in the trade-off between changes in objective and constraint values by providing a net measure of profitability.

In summary, I have made the following points:

1. if the continuous approximation has an integer optimum, then no duality gap prevails for GLM and both methods succeed in finding an optimum;

2. for linear contraints, GLM can produce the continuous approximation and there is a multiplier minimal for both Lagrangian duals;

3. in general, GLM has more tactical flexibility to provide integer base points from which a search may proceed.

In conclusion let us consider the case of a single, linear constraint more closely. I shall prove that the feasible GLM solution is better than the truncated CA solution for the special program:

$\text{Max} \sum_j f_j(x_j) : \sum_j a_j x_j \leq b,\ x \geq 0$ and x integer, where each f is monotone nondecreasing and $a > 0$.

At a minimal multiplier, y*, the GLM solution, $\{x^i\}_0^k$, can be ordered so that

$$\sum a_j x_j^0 \leq \ldots \leq \sum a_j x_j^r < b < \sum a_j x_j^{r+1} \leq \ldots \leq \sum a_j x_j^k,$$

where we have assumed a gap.

The multiplier, y*, is also minimal for the CA dual, C, since the g-constraint is linear. (This is unique if f is strictly concave.) Each CA solution, $\tilde{x}$, is not integer since otherwise GLM would have no gap. Let $\tilde{x}$ be feasible truncation of $\bar{x}$ (i.e., $\tilde{x} \leqq \bar{x}$ and $\tilde{x}$ is feasible), and let us show $f(x^r) \geqq f(\tilde{x})$

Since x_j^r solves

$$\text{Max } f_j\ (x_j) - y^*\ x_j\ :\ x_j \text{ integer, and } x_j \geqq 0,$$

it follows that

$$f_j(x_j^r) \geqq f_j(\tilde{x}_j) - y^*\ a_j\ (\tilde{x}_j - x_j^r).$$

Further, $\tilde{x} \leqq \bar{x}$, so

$$\sum f_j(x_j^r) \geqq \sum f_j(\tilde{x}_j) - y^* \sum a_j\ (\bar{x}_j - x_j^r).$$

However,

$$\sum a_j\ \bar{x}_j = b$$

implies

$$f(x^r) \geqq f(\tilde{x}) - y^*\ (b - \sum a_j\ x_j^r)$$

Since x^r is feasible, it follows that

$$f(x^r) \geqq f(\tilde{x}),$$

which is the desired result.

Note that judicious choice of the truncation, $\tilde{x}$, gives the GLM solution, which is the best possible rounding. The "judicious" choice is the inherent one when using GLM. Notice that x^0 corresponds to complete truncation, but x^r performed some upward rounding to get close to b.

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