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NO ABELIAN SEMIGROUP OPERATION IS COMPLETE

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Abstract

This paper shows that there does not exist a finite abelian semigroup <S, *> with order ≥ 3 such that the semigroup operation is complete over S. Neither is {*} complete with constants over S. J. C. Muzio has shown that over any finite space there exists a set of two abelian semigroup operations which is complete with constants over the space. Hence Muzio's result is best possible.

J. C. Muzio has shown that for any finite space $E(k) = \{1, 2, \ldots, k-1\}$, there exists a set $\{+, J\}$ of two operators such that each operator defines an abelian semigroup over $E(k)$ and the set $\{+, J\}$ is complete with constants over $E(k)$. [1] In this paper we show that there does not exist a finite abelian semigroup $\langle S, * \rangle$ whose operation is complete over S. Neither does there exist a finite abelian semigroup whose operation is complete with constants over S. Throughout we use the definitions and notation of [1]. We write "xy" for "x*y".

If S is an abelian semigroup and I is a subset of S, then I is an ideal of S if $SI \subset I$. An ideal I is semiprime if $x^2 \varepsilon I$ implies $x \varepsilon I$. An ideal I is prime if S-I is closed. [2, pp. 5, 71]

If $\sim$ is an equivalence relation on S, then * preserves the equivalence relation if $x \sim y$ and $z \sim w$ implies that $xz \sim yw$. Note that if $|S| \geq 3$ and if both $S = A \cup B$ and $A \cap B = \emptyset$, where A and B are both non-empty, then the decomposition $S = A \cup B$ induces a non-trivial, non-universal equivalence relation on S.

<u>Lemma 1</u>:- If S is a finite abelian semigroup and if I is a maximal, non-trivial ideal of S, and if for all $x \varepsilon S$ we have $x^2 \varepsilon I$, then $|S-I| = 1$.

<u>proof</u>:- Let $J = S-I$ and suppose the theorem is false, that is, suppose $|J| \geq 2$. Let $a \varepsilon J$. There are three cases.

Case 1: $aJ \subset I$. $I \cup \{a\}$ is an ideal of S. Since $|J| \geq 2$ $I \cup \{a\}$ is non-trivial. This contradicts the maximality of I in S.

Case 2: $aJ = J$. Since $a \varepsilon J$, $a \varepsilon aJ$, that is, there exists $y \varepsilon J$ such that $a = ay$. Multiplying by y we have $ay = ay^2$. Thus $y^2 \varepsilon I$ implies $a \varepsilon I$, which contradicts $a \varepsilon J$.

Case 3: $aJ \neq J$, $aJ \not\subset I$. $I \cup aJ$ is an ideal in S. Since $aJ \cup I$, I is not maximal. Since $aJ \neq J$, $I \cup aJ$ is not trivial. Thus I is not a maximal, non-trivial ideal in S.

Lemma 2;- If S is a finite abelian semigroup and if I is a maximal non-trivial ideal of S and if I is not semiprime, then $S \neq S^2$.

proof:- Let $J = S-I$. Let $A = \{x \mid x \varepsilon J, x^2 \varepsilon I\}$. $I \cup A$ is an ideal in S, for if $a \varepsilon A$ and $x \varepsilon S$, then $(ax)^2 = a^2x^2 \varepsilon I$, which implies that $ax \varepsilon A$. If A is empty, then I is semiprime, which is a contradiction. If A is non-empty, then $I \cup A$ is trivial, since I is known to be the maximal non-trivial ideal. Hence $J = A$. By Lemma 1, $|J| = 1$. But $a \varepsilon A$ implies $a^2 \varepsilon I$, that is, $a^2 \notin J$. Hence $a \notin S^2$.

The main theorem depends on the following result of Ivo Rosenberg: If A is a set of functions over a finite space S and if $\sim$ is a non-trivial, non-universal equivalence relation on S, for A to be complete it is necessary that A contain a function which does not preserve $\sim$. [3]

Theorem:- If <S, *> is a finite abelian semigroup but not a group, then the set {*} is not complete over S.

proof: If S is not a group then it contains a maximal, non-trivial ideal, say I. There are two cases.

Case 1: I is semiprime. Since I is maximal, I semiprime implies that I is prime, that is, that S-I is closed. [2, p.71] Let $J = S-I$. The decomposition $S = I \cup J$ induces an equivalence relation on S. Since I is an ideal, $II \subset I$, $IJ = JI \subset I$.

Since J is closed, $JJ \subset J$. The equivalence relation $\sim$ is preserved by the operation *.

Case 2: I is not semiprime. By Lemma 2, $S^2 \neq S$. Let $J = S - S^2$. The decomposition $S = S^2 \cup J$ induces an equivalence relation $\sim$ on S. Since each of the sets $(S^2)^2$, $S^2 J$, and J^2 is in S^2, the operation * preserves the equivalence relation $\sim$. Hence {*} is not complete.

Corollary 1:- If <S, *> is a finite abelian semigroup but not a group, then the set {*} is not complete with constants over S.

proof:- We need only note that each constant function preserves every equivalence relation over S.

Corollary 2:- If <S, *> is a finite abelian semigroup, then the set {*} is not complete with constants over S.

proof:- The author has proved elsewhere that the group operation of a finite abelian group is not complete with constants. [4, p. 396]

Since it is known that a nonabelian simple group is complete with constants [5] the abelian restriction cannot, in general, be removed.

References

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