Technical Report CS74007-R

Definability of Boolean Function Over Many-value Boolean Algebra

Andy N.C. Kang

and

Stewart N.T. Shen

January 1974

Computer Science Department

Virginia Polytechnic Institute and State University

Blacksburg, Virginia 24061

CR categories: 5.20, 5.21, 5.29
Keywords and Phrases: boolean algebra, boolean functions, definability

Abstract

Let B_s be the boolean algebra of s atoms and let $E(x_1, x_2, \ldots, x_n)$ be a boolean expression in n variables. We say that a function $f: B_s^n \to B_s$ is definable over B_s if there is a boolean expression $E(x_1, x_2, \ldots, x_n)$ such that for every n-tuple $(a_1, a_2, \ldots, a_n) \in B_s^n$, $f(a_1, a_2, \ldots, a_n) = E(a_1, a_2, \ldots, a_n)$. In this case, we say that the boolean expression $E(x_1, x_2, \ldots, x_n)$ defines a boolean function f.

In this paper, the definability of functions over B_s is first briefly discussed. We then give necessary and sufficient conditions on the definable functions over B_2, boolean algebra of four values. An efficient algorithm is also presented for finding the defining boolean expressions for the definable functions over B_2. The result is then extended to the functions over B_s.

I. Introduction:

Let $B_s = \langle B_s, +, \cdot, -, 0, 1\rangle$ be a boolean algebra of s atoms. Boolean functions in one variable from B_s to B_s are inductively defined in [2] and in which a characterization for them is pointed out. Boolean functions in n variables from B_s^n to B_s are defined in terms of boolean expressions, the elements of the free boolean algebra with n generators $x_1, x_2, \ldots, x_n$,[1,3] and the definition is given below.

Definition 1: A function $f\colon B_s^n \to B_s$ is definable over B_s if there is a boolean expression $E(x_1, x_2, \ldots, x_n)$ such that for every n-tuple $(a_1, a_2, \ldots, a_n) \in B_s^n$, $f(a_1, a_2, \ldots, a_n) = E(a_1, a_2, \ldots, a_n)$. In this case, f is called a boolean function defined by E.

It is known that when s is 1, each function from $B_1^n \to B_1$ is a boolean function. An algorithm to derive the defining boolean expression is also available.[2,3] However, the number of functions from B_s^n to B_s is $2^{s2^{sn}}$, while the number of boolean expressions in n variables is 2^{2^n}. Hence, when $s > 1$ there are functions which are not definable over B_s.

If a function $f\colon B_s^n \to B_s$ is definable, then there is a boolean expression which defines f. Each fundamental product in the disjunct normal form of the defining boolean expression defines a function that assumes a value comparable but less than the value of f. And the sum of the values of the functions defined by the fundamental products must agree with the function value of f. Based on this observation, the following theorem gives an algorithm to detect the definability of a function f. And in the case that f is definable, the algorithm will produce the boolean expression which defines f.

Theorem 1: A function f: $B_s^n \to B_s$ is definable over B_s if and only if the following algorithm gives an affirmative answer.

Step 1: list all the 2^n fundamental products in n variables, and then evaluate their associated function values over B_s.

Step 2: choose those fundamental products such that their associated function values are comparable and less than the function value of f. If and when the summation of the values of the chosen fundamental products is equal to the value of f, then the algorithm gives an affirmative answer. The disjunction of the chosen fundamental products is the defining expression for f. [x]

The proof of Theorem 1 is obvious. The theorem gives a straightforward answer to the definability problem. However, the algorithm presented by the theorem is not efficient. We shall further analyze the properties of the boolean functions over B_2, derive a characterization for them, and provide an efficient algorithm to generate the defining expressions. The results are then extended to B_s in general.

II. Definability of f over B_2

Consider the boolean algebra of four elements $B_2 = \{0, a, b, 1\}$, in which a and b are complementary to each other, 0 is the least element, and 1 is the greatest element.

We consider functions from B_2^n to B_2. An element in the domain of the functions is an n-tuple $\langle i_1, i_2, \ldots, i_n \rangle$ in which $i_j \in B_2$ for $1 \leq j \leq n$. We can represent a function by a list of (n + 1) - tuples with the rightmost coordinate containing the values of the function. The list representing a function is arranged in the standard order defined below.

Definition 2: A list of $(n + 1)$-tuples representing a function is called in standard order if the leftmost n-tuples are listed in the column major order (i.e. the leftmost coordinate varies most frequently).

Let us take a boolean expression $E_1(x_1, x_2, \ldots, x_n)$ and list the function f_1 defined by it in standard order. Then we define a new boolean expression E_i, $E_i(x_1, x_2, \ldots, x_n) = E_1(x_i, x_1, \ldots, x_{i-1}, \ldots, x_n)$, and also list the function f_i defined by E_i in standard order. We will see that the function values of f_i are just a rearrangement of the function value of f_1. This is immediately observed since E_i is obtained from E_1 by permutating the arguments x_1 by x_i, x_2 by $x_1, \ldots, x_i$ by x_{i-1}. To enumerate the values of f_i, consider constructing a "dummy" list of n+1 tuples which is obtained by rearranging the rows of the list of f_1 such that the ith coordinate varies most frequently. The last coordinate of the resulting list contains the function values of f_i in the order that they should appear in the standard order list of f_i. Notice that after the rearrangement, among the other n-1 coordinates the left coordinates vary more frequently than the right ones.

Since we are interested in the function values but not the arguments of f_i in standard order, we only need to copy the f_1 values into a new order and not really generate the dummy list mentioned above. Recall that the list of the ith argument of f_1 in standard order is $\overbrace{0 \text{ ---- } 0}^{4^{i-1}}$ $\overbrace{a \text{ --- } a}^{4^{i-1}}$ $\overbrace{b \text{ ---- } b}^{4^{i-1}}$ $\overbrace{1 \text{ ------- } 1}^{4^{i-1}}$ repeated 4^{n-i} times. Thus we can see that to obtain the f_i values in the desired order, we only need to record each of the 4^{n-i} continuous parts of the f_1 values in a specific way. To describe the way to record the f_1 values, let us first assume that the f_i values are already in the desired order and then consider the relationship among the f_i values and the f_1 values. Consider the f_i values in the desired order also in 4^{n-i} contiguous parts. In each part of the f_i values, the jth 4 contiguous elements are identical to the 4 elements taken every 4^{i-1} elements starting at the jth element in the corresponding part of the list of the f_1 values. That is, the kth ($k = 1,2,\ldots,4^i$) element in each part of the f_i values is identical to the pth element in the corresponding part of the f_1 values and $p = \lceil k/4 \rceil + 4^{i-1} (k - 4\lceil k/4 \rceil + 3)$. Where $\lceil m \rceil$ is the smallest integer not less than m.

Ignoring the partitioning of the f_i and f_1 values, we can see that the qth element of the f_i values in standard order is identical to the mth element of the f_1 values in standard order and $m = \lceil q/4 \rceil + 4^{i-1} (q - 4\lceil q/4 \rceil + 3\lceil \lceil q/4 \rceil / 4^{i-1} \rceil)$.

In the later discussions, we need to look at the 4-element contiguous groups of the f_i values and also some specific 4-element groups of the f_1 values. The following two definitions describe them formally.

Definition 3: Let f_1 be a function from $B_2^{\,n}$ to B_2 represented in standard order. Then the ith minor value set ($1 \le i \le n$) of f_1 is the set of all the contiguous 4-element function value groups in f_i where the qth element of the f_i values is identical to the mth element of the f_1 values, and

$$m = \lceil q/4 \rceil + 4^{i-1} (q - 4\lceil q/4 \rceil + 3\lceil \lceil q/4 \rceil / 4^{i-1} \rceil).$$

Definition 4: Let f be a function from $B_2^{\,n}$ to B_2 represented in standard order. Then the major value set of f is the set of the contiguous 4-contiguous-element function values associated with the argument groups whose values are of the form:

$$\begin{pmatrix} 0\ i_2 \ldots i_n \\ a\ i_2 \ldots i_n \\ b\ i_2 \ldots i_n \\ 1\ i_2 \ldots i_n \end{pmatrix}$$

Where i_j is either 0 or 1, $2 \le j \le n$.

Example 1: Suppose a function $f\colon B_2^{\,2} \to B_2$ represented in standard order has the function value set: <0ab1 00bb 0a0a 0000>

Then the major value set consists of $\begin{pmatrix}0\\a\\b\\1\end{pmatrix}$, and $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$. The first minor value set consists of $\begin{pmatrix}0\\a\\b\\1\end{pmatrix}$, $\begin{pmatrix}0\\0\\b\\b\end{pmatrix}$, $\begin{pmatrix}0\\a\\0\\a\end{pmatrix}$, and $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$. The second minor value set is composed of $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$, $\begin{pmatrix}a\\0\\a\\0\end{pmatrix}$, $\begin{pmatrix}b\\b\\0\\0\end{pmatrix}$, and $\begin{pmatrix}1\\b\\a\\0\end{pmatrix}$. ☒

Based on these definitions, we have the following results.

Lemma 1: If a function $f: B_2^n \to B_2$ is definable over B_2, then an element in the major value set is one of the following:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\a\\b\\1\end{pmatrix}, \begin{pmatrix}1\\b\\a\\0\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

Proof:

Assume $f: B_2^n \to B_2$ is defined by $E(x_1, x_2, \ldots, x_n)$. Since we are just considering the major value set, the argument value assigned to x_1 in E is $\begin{pmatrix}0\\a\\b\\1\end{pmatrix}$, and the argument values assigned to $x_2, x_3, \ldots, x_n$ in E are either $\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ or $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$.

The function value associated with $E(x_1, x_2, \ldots x_n)$ is generated by operations "OR", "AND", and "NOT" on the given arguments. Observing the following figure (Figure 1), we deduce that the possible outcome is one of the asserted elements. ☒

	NOT
0 0 0 0	1 1 1 1
0 a b 1	1 b a 0
1 b a 0	0 a b 1
1 1 1 1	0 0 0 0

0 a b 1	0 a b 1	0 0 0 0				
1 b a 0	1 b a 0	0 0 0 0	1 1 1 1	0 0 0 0		
1 1 1 1	1 1 1 1	0 0 0 0	1 1 1 1	0 a b 1	1 1 1 1	1 b a 0
OR / AND	0 0 0 0		0 a b 1		1 b a 0	

Figure 1

Lemma 2: If a function f: $B_2^n \to B_2$ is definable over B_2, then an element in any minor value set of f is one of the following:

$$\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\a\\0\\a\end{pmatrix}, \begin{pmatrix}0\\0\\b\\b\end{pmatrix}, \begin{pmatrix}0\\a\\b\\1\end{pmatrix}, \begin{pmatrix}a\\0\\a\\0\end{pmatrix}, \begin{pmatrix}a\\a\\a\\a\end{pmatrix}, \begin{pmatrix}a\\0\\1\\b\end{pmatrix}, \begin{pmatrix}a\\a\\1\\1\end{pmatrix},$$

$$\begin{pmatrix}b\\b\\0\\0\end{pmatrix}, \begin{pmatrix}b\\1\\0\\a\end{pmatrix}, \begin{pmatrix}b\\b\\b\\b\end{pmatrix}, \begin{pmatrix}b\\1\\b\\1\end{pmatrix}, \begin{pmatrix}1\\b\\a\\0\end{pmatrix}, \begin{pmatrix}1\\1\\a\\a\end{pmatrix}, \begin{pmatrix}1\\b\\1\\b\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

Proof:

We assume f is represented in standard order and is defined by $E(x_1, x_2, \ldots, x_n)$. An element of the first minor value set of f (that is, the set of function values in the consecutive 4-element groups) is the outcome of operations "OR", "AND", and "NOT: performed on the arguments

$$\begin{pmatrix}0\\a\\b\\1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}a\\a\\a\\a\end{pmatrix}, \begin{pmatrix}b\\b\\b\\b\end{pmatrix}, \text{ and } \begin{pmatrix}1\\1\\1\\1\end{pmatrix}.$$

This is because when f is represented in standard order, the leftmost coordinate alternates in 0, a, b, 1 and, in general, the jth coordinate alternates in $\overbrace{0 \ldots 0}^{4^{j-1}}$, $\overbrace{a \ldots a}^{4^{j-1}}$, $\overbrace{b \ldots b}^{4^{j-1}}$, $\overbrace{1 \ldots 1}^{4^{j-1}}$, repeated 4^{n-j} times. By exhausting all the possible operations on the given argument sets, we have the following figure (Figure 2), in which we observe that any possible outcome is one of the asserted elements.

Note that the ith minor value set of f can be considered as the first minor value set of f_i where f_i is defined by $E(x_i, x_1, \ldots, x_{i-1}, x_{i-1}, \ldots, x_n)$. Thus, we have proved the lemma. ☒

Figure 2

1	1	0																												
1	1	0																												
1	1	0																												
1	1	0																												
a	a	0	1	a																										
a	a	0	1	a																										
a	a	0	1	a																										
a	a	0	1	a																										
b	b	0	1	b	1	0																								
b	b	0	1	b	1	0																								
b	b	0	1	b	1	0																								
b	b	0	1	b	1	0																								
0	0	0	1	0	a	0	b	0																						
a	a	0	1	a	a	a	1	0																						
b	b	0	1	b	1	0	b	b																						
1	1	0	1	1	1	a	1	b																						
1	1	0	1	1	1	a	1	b	1	0																				
b	b	0	1	b	1	0	b	b	1	0																				
a	a	0	1	a	a	a	1	0	1	0																				
0	0	0	1	0	a	0	b	0	1	0																				
a	a	0	1	a	a	a	1	0	a	0	1	a																		
a	a	0	1	a	a	a	1	0	a	a	1	0																		
1	1	0	1	1	1	a	1	0	1	b	1	a																		
1	1	0	1	1	1	a	1	0	1	1	1	0																		
b	b	0	1	b	1	0	b	b	b	0	1	b	1	0																
b	b	0	1	b	1	0	b	b	1	0	b	b	1	0																
0	0	0	1	0	a	0	b	0	b	0	a	0	1	0																
0	0	0	1	0	a	0	b	0	1	0	0	0	1	0																
0	0	0	1	0	a	0	b	0	0	0	1	0	a	0	b	0														
a	a	0	1	a	a	a	1	0	a	a	1	0	a	a	1	0														
0	0	0	1	0	a	0	b	0	b	0	a	0	1	0	0	0														
a	a	0	1	a	a	a	1	0	1	a	a	0	1	a	a	0														
1	1	0	1	1	1	a	1	b	1	0	1	1	1	a	1	b	1	0												
b	b	0	1	b	1	0	b	b	1	0	b	b	1	0	b	b	1	0												
1	1	0	1	1	1	a	1	b	1	b	1	a	1	1	1	0	1	0												
b	b	0	1	b	1	0	b	b	1	b	b	0	1	b	b	0	1	0												
b	b	0	1	b	1	0	b	b	b	0	1	b	1	0	b	b	b	0	1	b										
1	1	0	1	1	1	a	1	b	1	a	1	b	1	a	1	b	1	a	1	b										
b	b	0	1	b	1	0	b	b	b	b	1	0	1	0	b	0	b	0	1	b										
1	1	0	1	1	1	a	1	b	1	1	1	0	1	1	1	0	1	a	1	b										
a	a	0	1	a	a	a	1	0	a	0	1	a	a	a	1	0	a	0	1	a	1	0								
0	0	0	1	0	a	0	b	0	a	0	b	0	a	0	b	0	a	0	b	0	1	0								
a	a	0	1	a	a	a	1	0	1	0	a	a	1	a	a	0	a	0	1	a	1	0								
0	0	0	1	0	a	0	b	0	1	0	0	0	1	0	0	0	a	0	b	0	1	0								
0	0	0	1	0	a	0	b	0	0	0	1	0	a	0	b	0	0	0	1	0	b	0	a	0						
0	0	0	1	b	a	0	b	0	a	0	b	0	a	0	b	0	a	0	b	0	1	0	0	0						
b	b	0	1	b	1	0	b	b	b	b	1	0	1	0	b	0	b	0	1	b	b	b	1	0						
b	b	0	1	b	1	0	b	b	1	b	b	0	1	0	b	0	1	0	b	b	1	b	b	0						
1	1	0	1	1	1	a	1	b	1	0	1	1	1	a	1	b	1	0	1	1	1	b	1	a	1	0				
1	1	0	1	1	1	a	1	b	1	a	1	b	1	a	1	b	1	a	1	b	1	1	1	0	1	0				
a	a	0	1	a	a	a	1	0	1	0	a	a	1	a	a	0	a	0	1	a	1	0	a	a	1	0				
a	a	0	1	a	a	a	1	0	1	a	a	0	1	a	a	0	a	a	1	0	1	a	a	0	1	0				
a	a	0	1	a	a	a	1	0	a	0	1	a	a	a	1	0	a	0	1	a	1	0	a	a	a	0	1	a		
0	0	0	1	0	a	0	b	0	a	0	b	0	a	0	b	0	a	0	b	0	1	0	0	0	0	0	1	0		
1	1	0	1	1	1	a	1	b	1	b	1	a	1	1	1	0	1	0	1	1	1	b	1	a	1	b	1	a		
b	b	0	1	b	1	0	b	b	1	b	b	0	1	b	b	0	1	0	b	b	1	b	b	0	b	b	1	0		
b	b	0	1	b	1	0	b	b	b	0	1	b	1	0	b	b	b	0	1	b	b	b	1	0	b	0	1	b	1	0
1	1	0	1	1	1	a	1	b	1	a	1	b	1	a	1	b	1	a	1	b	1	1	1	0	1	0	1	1	1	0
0	0	0	1	0	a	0	b	0	b	0	a	0	1	0	0	0	0	0	1	0	b	0	a	0	b	0	a	0	1	0
a	a	0	1	a	a	a	1	0	1	a	a	0	1	a	a	0	a	a	b	0	1	a	a	0	1	0	a	a	1	0
OR	0		1		a		b		0		1		a		b		0		1		b		a		0		1		a	
	0		1		a		b		a		b		a		b		a		b		1		0		0		1		0	
AND	0		1		a		b		b		a		1		0		0		1		b		a		b		a		1	
	0		1		a		b		1		0		1		0		a		b		1		0		b		a		b	

Lemma 3: Let f be a function from B_2^n to B_2. Assume every element in the major value set of f is given and satisfies Lemma 1. Then the function is unique if every element in all the minor value sets of f satisfies Lemma 2.

Proof:

Assume only the major value set of f has been given and it satisfies Lemma 1. Other function values are to be assigned. Then to make the second minor value set of f satisfy Lemma 2, we have to assign properly the unspecified function values associated with the n-tuple arguments of the forms:

$$\begin{pmatrix} 0 & a & i_3 & \cdots & i_n \\ a & a & i_3 & \cdots & i_n \\ b & a & i_3 & \cdots & i_n \\ 1 & a & i_3 & \cdots & i_n \end{pmatrix} \text{ and } \begin{pmatrix} 0 & b & i_3 & \cdots & i_n \\ a & b & i_3 & \cdots & i_n \\ b & b & i_3 & \cdots & i_n \\ 1 & b & i_3 & \cdots & i_n \end{pmatrix}$$

where i_j is either 0 or 1 for $3 \le j \le n$.

To satisfy Lemma 2, if the top and the bottom rows of a minor value set element are given, then the middle two rows of that element have to be unique. After the major value set has been chosen, there is only one way to make certain elements of the second minor value set satisfy Lemma 2 because that the top and the bottom rows of these elements of the second minor value set have been defined through the major value set. Similarly, there is always only one way to assign some unspecified function values to make some elements of the ith minor value set satisfy Lemma 2, once we have properly assigned all the values we can in the i-1th minor value set. After we have assigned all the values to the function to make all the minor value sets satisfy Lemma 2, we have specified all the function values. That is, the function is unique. ☒

The following example illustrates the process described in Lemma 3.

Example 2: Let f be a function from $B_2^{\,3}$ to B_2. We assume that only the major value set has been assigned. The function looks like this:

x_1	x_2	x_3	f
0	0	0	0
a	0	0	a
b	0	0	b
1	0	0	1
0	a	0	?
a	a	0	?
b	a	0	?
1	a	0	?
0	b	0	?
a	b	0	?
b	b	0	?
1	b	0	?
0	1	0	0
a	1	0	0
b	1	0	0
1	1	0	0

x_1	x_2	x_3	f
0	0	a	?
a	0	a	?
b	0	a	?
1	0	a	?
0	a	a	?
a	a	a	?
b	a	a	?
1	a	a	?
0	b	a	?
a	b	a	?
b	b	a	?
1	b	a	?
0	1	a	?
a	1	a	?
b	1	a	?
1	1	a	?

x_1	x_2	x_3	f
0	0	b	?
a	0	b	?
b	0	b	?
1	0	b	?
0	a	b	?
a	a	b	?
b	a	b	?
1	a	b	?
0	b	b	?
a	b	b	?
b	b	b	?
1	b	b	?
0	1	b	?
a	1	b	?
b	1	b	?
1	1	b	?

x_1	x_2	x_3	f
0	0	1	1
a	0	1	1
b	0	1	1
1	0	1	1
0	a	1	?
a	a	1	?
b	a	1	?
1	a	1	?
0	b	1	?
a	b	1	?
b	b	1	?
1	b	1	?
0	1	1	1
a	1	1	b
b	1	1	a
1	1	1	0

After the unique value assignments of some elements in the second minor value set to satisfy Lemma 2, we have:

x_1	x_2	x_3	f
0	0	0	0
a	0	0	a
b	0	0	b
1	0	0	1
0	a	0	0
a	a	0	0
b	a	0	b
1	a	0	b
0	b	0	0
a	b	0	a
b	b	0	0
1	b	0	a
0	1	0	0
a	1	0	0
b	1	0	0
1	1	0	0

x_1	x_2	x_3	f
0	0	a	?
a	0	a	?
b	0	a	?
1	0	a	?
0	a	a	?
a	a	a	?
b	a	a	?
1	a	a	?
0	b	a	?
a	b	a	?
b	b	a	?
1	b	a	?
0	1	a	?
a	1	a	?
b	1	a	?
1	1	a	?

x_1	x_2	x_3	f
0	0	b	?
a	0	b	?
b	0	b	?
1	0	b	?
0	a	b	?
a	a	b	?
b	a	b	?
1	a	b	?
0	b	b	?
a	b	b	?
b	b	b	?
1	b	b	?
0	1	b	?
a	1	b	?
b	1	b	?
1	1	b	?

x_1	x_2	x_3	f
0	0	1	1
a	0	1	1
b	0	1	1
1	0	1	1
0	a	1	1
a	a	1	b
b	a	1	1
1	a	1	b
0	b	1	1
a	b	1	1
b	b	1	a
1	b	1	a
0	1	1	1
a	1	1	b
b	1	1	a
1	1	1	0

After the unique value assignments of some elements in the third minor value set to satisfy Lemma 2, we have:

x_1	x_2	x_3	f
0	0	0	0
a	0	0	a
b	0	0	b
1	0	0	1
0	a	0	0
a	a	0	0
b	a	0	b
1	a	0	b
0	b	0	0
a	b	0	a
b	b	0	0
1	b	0	a
0	1	0	0
a	1	0	0
b	1	0	0
1	1	0	0

x_1	x_2	x_3	f
0	0	a	a
a	0	a	a
b	0	a	1
1	0	a	1
0	a	a	a
a	a	a	0
b	a	a	1
1	a	a	b
0	b	a	a
a	b	a	a
b	b	a	a
1	b	a	a
0	1	a	a
a	1	a	0
b	1	a	a
1	1	a	0

x_1	x_2	x_3	f
0	0	b	b
a	0	b	1
b	0	b	b
1	0	b	1
0	a	b	b
a	a	b	b
b	a	b	b
1	a	b	b
0	b	b	b
a	b	b	1
b	b	b	0
1	b	b	a
0	1	b	b
a	1	b	b
b	1	b	0
1	1	b	0

x_1	x_2	x_3	f
0	0	1	1
a	0	1	1
b	0	1	1
1	0	1	1
0	a	1	1
a	a	1	b
b	a	1	1
1	a	1	b
0	b	1	1
a	b	1	1
b	b	1	a
1	b	1	a
0	1	1	1
a	1	1	b
b	1	1	a
1	1	1	0

We can see that the function is unique. Utilizing the previous lemmas, we can prove the theorem below which gives the necessary and sufficient conditions on definable functions over B_2.

<u>Theorem 2</u>: Let f be a function from B_2^n to B_2. f is definable over B_2 if and only if the list of f in standard order has the following two properties:

(1) The major value set of f satisfies Lemma 1.

(2) The minor value sets of f satisfy Lemma 2.

<u>Proof</u>:

The necessary condition follows directly from Lemma 1 and Lemma 2. The proof that this is also a sufficient condition follows.

The n-tuple argument set corresponding to an element of the major value set is of the form:

$$\begin{pmatrix} 0 & i_2 & \cdots & i_n \\ a & i_2 & \cdots & i_n \\ b & i_2 & \cdots & i_n \\ 1 & i_2 & \cdots & i_n \end{pmatrix}$$

where i_j is either 0 or 1 for $2 \le j \le n$. Thus, we know that there are 2^{n-1} distinct argument sets which may go with an element of the major value set. In addition, an element in a major value set can be arbitrarily chosen from one of the four values given in Lemma 1. Thus, there are $4^{2^{n-1}} = 2^{2^n}$ different ways to assign a major value set to satisfy Lemma 1. By Lemma 3, a function satisfying the two properties is uniquely determined by assigning a major value set. Therefore, we deduce that there are 2^{2^n} functions which satisfy the two properties.

In n variables there are 2^{2^n} different boolean expressions. Each boolean expression defines a boolean function. Thus, there are 2^{2^n} different definable boolean functions. All these functions satisfy properties (1) and (2) by Lemma 1 and Lemma 2. From the fact that there are exactly 2^{2^n} functions which satisfy both properties, therefore we deduce that each boolean expression must define one of them and vice versa.

[x]

III. Deriving the Defining Boolean Expressions

The preceding theorem leads to the following algorithm with which we can find the defining boolean expression efficiently. The algorithm has two steps. The first step applies the theorem to detect the definability of a function. The second step is to construct the defining boolean expressions for any definable function. The algorithm in the second step makes use of the fact that a major value set uniquely determines a definable function.

Algorithm: Given a function f: $B_2^n \to B_2$. List f in standard order.

Step 1: See if f satisfies the two properties in theorem 2. If not, f is not definable over B_2. Otherwise, go to step 2.

Step 2: The n-tuple argument corresponding to an element in the major value set is

$$\begin{pmatrix} 0 & i_2 & \dots & i_n \\ a & i_2 & \dots & i_n \\ b & i_2 & \dots & i_n \\ 1 & i_2 & \dots & i_n \end{pmatrix}$$

where i_j are 0 or 1 for $2 \le j \le n$. Let

$$x^i = \begin{cases} x & \text{if } i = 1 \\ \bar{x} & \text{if } i = 0. \end{cases}$$

Start with $E(x_1, x_2, \ldots, x_n) = 0$. For each of the 2^{n-1} elements in the major value set, do one of the following:

Case 1: if the element is $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, do nothing.

Case 2: if the element is $\begin{pmatrix} 0 \\ a \\ b \\ 1 \end{pmatrix}$, then $E(x_1, x_2, \ldots, x_n) \leftarrow E(x_1, x_2, \ldots, x_n)$ $+ x_1 x_2^{i_2} x_3^{i_3} \ldots x_n^{i_n}$, where i_j's are from the corresponding n-tuple arguments.

Case 3: if the element is $\begin{pmatrix} 1 \\ b \\ a \\ 0 \end{pmatrix}$, then $E(x_1, x_2, \ldots, x_n) \leftarrow E(x_1, x_2, \ldots, x_n)$ $+ \bar{x}_1 x_2^{i_2} x_3^{i_3} \ldots x_n^{i_n}$, where i_j's are from the corresponding n-tuple arguments.

Case 4: if the element is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, then $E(x_1, x_2, \ldots, x_n) \leftarrow E(x_1, x_2, \ldots, x_n)$ $+ x_2^{i_2} x_3^{i_3} \ldots x_n^{i_n}$, where i_j's are from the corresponding n-tuple arguments. ☒

Proof of the Algorithm:

Step 1 of the algorithm is true according to Theorem 2.

Now let us prove step 2. It is obvious that the resultant expression satisfies the major value set of f. Hence, this expression must also satisfy all the minor value sets of f because the function is definable. So we have the conclusion that the expression derived according to this algorithm defines the given definable function. ☒

Example 3: Let f be the function defined in example 2. We know it is definable. let us look at the part of the list corresponding to the major value set:

$$\begin{pmatrix} a_1 & i_2 \cdots i_n \\ a_2 & i_2 \cdots i_n \\ & \vdots \\ a_{2^s} & i_2 \cdots i_n \end{pmatrix}$$

where i_j is either a_1 or a_{2^s} for $2 \le j \le n$.

<u>Definition 6</u>. Let f be a function from $B_s^{\ n}$ to B_s represented in its standard order. Then the <u>ith minor value set</u> of f $(1 \le i \le n)$ is the set of the $2^{s(n-1)}$ contiguous 2^s-element function value groups from a new list of function values which is obtained by copying the mth f value as the qth element of the new list and $m = \lceil q/2^s \rceil + 2^{s(i-1)} (q - 2^s \lceil q/2^s \rceil + (2^s - 1) \lceil \lceil q/2^s \rceil / 2^{s(i-1)} \rceil)$.

If a function is definable then it is defined in terms of an expression $E(x_1, x_2, \ldots, x_n)$. The major value set associated with E is generated by performing "AND", "OR", and "NOT" on the set of arguments of the form:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2^s} \end{pmatrix}, \begin{pmatrix} a_1 \\ a_1 \\ \vdots \\ a_1 \end{pmatrix}, \text{ and } \begin{pmatrix} a_{2^s} \\ a_{2^s} \\ \vdots \\ a_{2^s} \end{pmatrix}.$$

From the fact that $\bar{a}_i = a_{2^s - i + 1}$ for $i = 1, 2, \ldots, 2^{s-1}$ and $\bar{a}_j \cdot a_j = a_1$, $\bar{a}_j + a_j = a_{2^s}$ for $j = 1, 2, \ldots, 2^s$, we see that the possible outcome from these operations is one of the following four elements:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2s} \end{pmatrix}, \begin{pmatrix} a_{2^s} \\ a_{2^s - 1} \\ \vdots \\ a_1 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_1 \\ \vdots \\ a_1 \end{pmatrix}, \text{ and } \begin{pmatrix} a_{2^s} \\ a_{2^s} \\ \vdots \\ a_{2^s} \end{pmatrix}. \qquad \text{(A)}$$

Similarly, for deriving the minor value sets, if we perform "AND", "OR", and "NOT" on the set of arguments of the form:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2^s} \end{pmatrix}, \begin{pmatrix} a_1 \\ a_1 \\ \vdots \\ a_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ a_2 \\ \vdots \\ a_2 \end{pmatrix}, \text{ and } \begin{pmatrix} a_{2^s} \\ a_{2^s} \\ \vdots \\ a_{2^s} \end{pmatrix},$$

we will derive 2^{2^s} different possible outcomes of the following forms:

$$\begin{pmatrix} a_i \\ \cdot \\ \cdot \\ \cdot \\ a_1 \end{pmatrix}, \begin{pmatrix} a_i \\ \cdot \\ \cdot \\ \cdot \\ a_2 \end{pmatrix}, \ldots, \text{ and} \begin{pmatrix} a_i \\ \cdot \\ \cdot \\ \cdot \\ a_{2^s} \end{pmatrix} \quad \text{for } i = 1,2,\ldots,2^s. \qquad \text{(B)}$$

Now, if a function has its major value set assigned, then there is a unique way to assign the function values so that certain elements of the second minor value set will fall into one of the forms in (B). This is again because the top and the bottom rows of these second minor value set elements have been defined by the major value set. Iterate the same procedure, the function will be uniquely determined if we make all the minor set elements fall into the given forms in (B).

Based on these arguments, a characterization theorem for definable boolean functions over B_s is given without proof as follows.

Theorem 3. Let f be a function from $B_s{}^n$ to B_s. f is definable over B_s if and only if the list of f in standard order has two properties:

(1) Each element in the major value set of f is in one of the forms in (A).

(2) Each element in the minor value sets of f is in one of the forms in (B).

☒

An algorithm to derive the defining expression for a definable function is an obvious extension of the previous algorithm. We only need to look at the major value set of a definable function. In the major value set, if an element is

$$\begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_{2^s} \end{pmatrix}, \begin{pmatrix} a_{2^s} \\ a_{2^s} \\ \cdot \\ \cdot \\ \cdot \\ a_{2^s} \end{pmatrix}, \text{ or} \begin{pmatrix} a_{2^s} \\ a_{2^s-1} \\ \cdot \\ \cdot \\ \cdot \\ a_1 \end{pmatrix},$$

then we will include a term $x_1 x_2^{i_2} \ldots x_n^{i_n}$, $x_2^{i_2} \ldots x_n^{i_n}$, or $x_1 x_2^{i_2} \ldots x_n^{i_n}$, respectively into the defining expression, where i_j's are in the corresponding n-tuple arguments.

V. Discussion

With a reasonably large s, it is difficult to list the 2^{2s} possible items in (B) in section IV. However, it is not necessary to list all these items. We can first check the validity of the major value set against the four items in (A) in Section IV and apply the second step of the algorithm to derive a boolean expression. Then we evaluate the boolean expression over B_s and compare the results with the function values of f. If they agree then f is definable and the defining expression is also derived, otherwise f is not definable.

Aknowledgements

The authors would like to thank R. Manor for helpful discussions.

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