On the Dynamic Response of Variable-Rate, Sampled-Data Systems

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Abstract

Time-domain analysis is used to derive criteria for the stability of sampled-data feedback systems comprising a time-varying plant, a time-varying non-linearity, and a sampler which may exhibit any given periodic or aperiodic sampling mode. Three aspects of the systems dynamic response are given a unified treatment: boundedness, unboundedness, and asynptotic decay. In addition to qualitative criteria, the results provide quantitative bounds on the system response, and indicate its explicit dependence on the sampling mode.

1. Introduction

Papers by Haddad [1]-[4] have presented some interesting results on the stability and dynamic response of nonlinear feedback systems containing a time-varying plant. In this paper, stability criteria are derived for discrete-time nonlinear feedback systems with a sampler at the input of the linear subsystem, as shown in Figure 1. The block H represents a time-varying linear system interacting with the time-varying nonlinear characteristic N via the sampler S_T. The sampler is not restricted to have a uniform sampling rate but may exhibit any mode of the sampling function or distribution of the sampling instants. Thus, the applicability of the analysis and criteria of this paper is not restricted to systems with the conventional uniform sampling schemes, but extends to systems employing *aperiodic, cyclic-variable, or skip sampling* techniques. Such sampling schemes are not uncommon and may be quite useful in practice. It has been shown that cyclic-variable rate sampling offers many useful features such as improvement of the performance of a sampled-data system and the time-sharing of equipment in discrete-data systems [6]. The skip sampling technique may be employed to obtain a novel type of controller with which dead-beat performance can be achieved.

The method of analysis presented in this paper is rather distinct from the types of analysis to be found in previous investigations of the stability problem. In the course of the derivation, the system behavior is examined entirely in the *time-domain,* with no reference to Liapunov functions, frequency-domain relationships, or normed spaces.

In the present approach, a unified treatment is adopted in the analysis of the different stability properties considered. The three aspects of stability behavior studied, namely *boundedness, unboundedness, and decay,* are interrelated by common mathematical criteria. In this respect, the present criteria provide an explicit connection between the '*internal*' and '*external*' stability behavior of the system Furthermore, these criteria show explicitly the dependence of the systems dynamic repsonse behavior on the sampling which could exhibit any uniform or nonuniform pattern.

A survey of the stability literature indicates that many of the significant results give only qualitative predictions about the dynamical behavior of the system. From practical and theoretical standpoints, it is equally important to predict *quantitative bounds* on the response of the system. The results of this paper, in addition to establishing sufficient conditions for boundedness, unboundedness and decay, also furnish bounds on the various system waveforms. The magnitude of such bounds may be regarded as a measure of the '*degree of stability*' of the system. One practical application for such bounds is in the estimation of the amplitude of limit-cycles or steady-state waveforms that might be developed by the system.

The interpretation of the results of this paper and their application to specific systems is fairly straightforwarrd. The conditions for stability stated in the criteria lend themselves to a simple *graphical interpretation.* Application of the results is illustrated by an example.

2. System Description, Assumptions, and the Stability Problem

In the feedback system of Figure 1, the linear time-varying subsystem H is characterized by its *impulse response* h(t, z), which is assumed to be an ordinary function of t and z containing no delta functions. The input-output relationship for H is

$$g(t) = \int_{t_0}^{t} u^*(z)h(t,z)dz, \qquad t \geq t_0 \tag{1}$$

where t_0 is the given initial instant of observation of the system. Ordinarily, the relationship in (1) would include on the right-hand side on additive term $g_0(t)$ which solely depends on the initial conditions existing in the plant at t_0 , and may therefore be described as the *unforced response* of H. It is assumed here that such a term is combined with the input $r_1(t)$. This entails no loss of generality , yet it allows H to be treated as a linear mapping from input u to the output g . Thus, one may write

$$g = H(u)^* \tag{2}$$

where H stands for the linear operator represented in (1).

The *ideal sampler* S_T at the input of the plant produces a sequence of impulses weighted by the values of $u(t)$ at the sampling instants. The *sampling instants* are not necessarily uniformly spaced but may have any desired distribution along the time axis. Let $T(t_0, t)$ be the set of sampling instants in the time interval $[t_0, t]$. The set $T(t_0, t)$ is a finite set of distinct real numbers arbitrarily distributed in the interval $[t_0, t]$. Thus one may write

$$T(t_0, t) = \{t_1, t_2, t_3, \ldots, t_m\}, \qquad \text{where} \quad t_0 \leq t_n \leq t \ , \ t_{n+1} > t_n$$

For any given system, the functional dependence of $T(t_0, t)$ on t_0 and t represents the sampling mode of the system. For the particular case where the sampling is uniform with period τ, one has

$$T(t_0, t) = \{ t_1, \ t_1 + \tau, \ t_1 + 2\tau, \ t_1 + 3\tau, \ \ldots\}$$

In general, the delta modulated output $u^*(t)$ of the sampler may be expressed as

$$u^*(t) = \sum_{z \in T(t_0, t)} u(z)\delta(t - z) \tag{3}$$

Any linear hold element that might typically follow the sampler is assumed to be lumped with the subsystem H. Combining (1) and (3), the input-output relationship for the cascade of sampler and plant is

$$g(t) = \sum_{z \in T(t_0, t)} u(z)h(t, z) \tag{4}$$

In operator notation, (4) may be written as

$$g = H(u)^* \tag{5}$$

The constraint induced by the block N is assumed to be a nonlinear time-varying relationship between $e(t)$ and $f(t)$ of the form

$$f(t) = n(e(t), t) \ \text{ for all } \ t \geq t_0 \tag{6}$$

The relationship in (6) need not represent a unique correspondence between the functions e and f. In other words, the relationship $y = n(x, t)$ is allowed to be *multivalued* in x. Thus, we are admitting a class of nonlinear characteristics that exhibit *hysteresis.* We assume that for any fixed t and any $x \in (-\infty, \infty)$ there is at least one value of $y \in (-\infty, \infty)$ that satisfies the relationship $y = n(x, t)$. Note that under these conditions, the constraint $y = n(x, t)$ does not always qualify as an operator or mapping from $(-\infty, \infty)$ into $(-\infty, \infty)$. No conditions of continuity are stipulated on the relationship $y = n(x, t)$. Furthermore, no sector restriction is imposed on the graph of $y = n(x, t)$, thereby admitting cases where $n(x, t)$ is *discontinuous* at $x = 0$ and $n(0, t) \neq 0$.

It is assumed that the system signals e, f, u, g, r_1, and r_2 are defined for all $t \geq t_0$, and that these signals are finite for finite values of t, i.e.,

$$|e(t)| < \infty, \text{ for } t < \infty$$

In other words, it is assumed that the system exhibits no "finite escape time."[1]

Three questions related to the stability and dynamic response of the sampled-data feedback system will be considered:

1. *Boundedness:* what conditions on the plant H, the nonlinearity N, and the inputs (r_1, r_2), would guarantee the system signals are bounded. Quantitative upper and lower *bounds* on these signals will be evaluated.
2. *Asymptotic decay:* what conditions ensure that the system signals tend to zero as t tends to infinity
3. *Divergence:* conditions that would force the system signals to be unbounded.

3. Preliminaries

In this section we introduce the special definitions and symbols to be used in the statement and derivation of the main results. A transformation involving two parameter gain functions, k(t) and $\beta(t)$, is introduced. This transformation may be thought of as representing a trade-off of time-varying gains between the non-linearity N and the linear plant H. For a given N, H, r_1, and r_2, any choice of the functions k(t) and $\beta(t)$ would produce a corresponding nonlinearity $N_{k\beta}$, a corresponding plant $H_{k\beta}$, and a single effective input r_k which are defined below. The stability conditions, to be discussed in a following section, will be expressed in terms of $r_k(t)$ and certain parameters obtained from the systems $H_{k\beta}$ and $N_{k\beta}$.

3.1 The systems $N_{k\beta}$, $H_{k\beta}$, $\hat{H}_{k\beta}$, and the effective input $r_k(t)$

Let k(t) and $\beta(t)$ be arbitrary real-valued functions defined for all t, with $\beta^{-1}(t)$ well-defined for all t. Consider the systems $N_{k\beta}$, $H_{k\beta}$, $\hat{H}_{k\beta}$ and R_k described in Figure 2. These systems are composed of the given nonlinearity N, the plant H, the sampler S_T, and the blocks designated k(t), $\beta(t)$, and $\frac{1}{\beta(t)}$ which represent the multiplicative time-varying gains of the transformation. The constraint induced by $N_{k\beta}$ between its terminatl signals e and y is

$$y = \beta[n(e, t) - ke] \qquad \text{for all } t \geq t_0 \qquad (7)$$

Hence $N_{k\beta}$ may be characterized by a non-linear relationship $N_{k\beta}$ defined as

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[1]For discussions and criteria on the problem of finite escape time, see references [6] thru [10].

$$N_{k\beta}(x, t) \equiv \beta[n(x, t) - kx] \tag{8}$$

such that

$$y = N_{k\beta}(e, t) \tag{9}$$

For the linear system $H_{k\beta}$ in Figure 2b, the governing relationship between the terminal signals w and g is

$$g = H\left(\frac{w}{\beta} - kg\right)^* \tag{10}$$

where H is the linear operator defined in (1) and (2), and * is the sampling operator. The system designated $H_{k\beta}$ in Figure 2c is identical to the system in Figure 2b, in so far as the input-output relationship is concerned. For the system in Figure 2c, this relationship is

$$g = H\left[\frac{w^*}{\beta} - (kg)^*\right] \tag{11}$$

Remembering that, for an ideal sampler, $\frac{w^*}{\beta} = \left(\frac{w}{\beta}\right)^*$ and $\left(\frac{w}{\beta}\right)^* - (kg)^* = \left(\frac{w}{\beta} - kg\right)^*$, the relationship in (11) becomes

$$g = H\left(\frac{w}{\beta} - kg\right)^* \tag{12}$$

which is identical to the input-output relationship (1) of the system $H_{k\beta}$ in Figure 2b. The linear system $\hat{H}_{k\beta}$ is equivalent to the system $H_{k\beta}$. Thus, in operator rotation, one has

$$g = H_{k\beta}(w) = \hat{H}_{k\beta}(w)^* \tag{13}$$

The linear system R_k shown in Figure 2d is defined in terms of the system H_{k1}, i.e., the system $H_{k\beta}$ with $\beta(t) = 1$. The system R_k maps r_1 and r_2 into the signal r_k according to the following relationship

$$r_k = R_k(r_1 r_2) = r_1 - H_{k1}(r_2 + kr_1) \tag{14}$$

The signal $r_k(t)$, which depends on the input $r_1(t)$, $r_2(t)$ and on $k(t)$, will be referred to as the system "effective input" corresponding to the specific choice of $k(t)$.

We now elaborate on the relationship among $N_{k\beta}$, $\hat{H}_{k\beta}$ and $r_k(t)$, and the given feedback system S. For a given pair of inputs (r_1, r_2) let e, u , g, and f be the corresponding outputs. For a given choice of the parameters (k, β) , define the function y as

$$y(t) = \beta(t)\,[f(t) - k(t)e(t)] \tag{15}$$

It is shown in the Appendix that the signal $y(t)$ thus defined, and the signal $e(t)$, satisfy the following relationships:

$$\text{(i)}\ y = n_{k\beta}(e, t) \qquad \text{(ii)}\ e = r_k - H_{k\beta}(y)^* \tag{16}$$

These two relationships imply that the signals e, y, and r_k may be associated with the system $S_{k\beta}$ shown in Figure 3. This sampled-data feedback system is composed of the linear system $\hat{H}_{k\beta}$, the non-linearity $N_{k\beta}$, and has a single input $r_k(t)$, Note that the signal $e(t)$ in the system $S_{k\beta}$ is identical to the signal e(t) in the original feedback system S of Figure 1. Thus, for any choice of the parameter functions [k(t), β(t)] one may transform the system S into the system $S_{k\beta}$ without affecting the signal e(t). Hence, one may examine the stability properties of the system S by analyzing the behavior of the signal e(t) in the system $S_{k\beta}$ for any choice of the gain functions k(t) and β(t).

Although the systems $\hat{H}_{k\beta}$, $N_{k\beta}$, and R_k are well-defined for an arbitrary k(t) and an arbitrary invertible β(t), we shall find it necessary for our purposes to restrict the choice of the pair (k, β) by a number of conditions. These conditions are designed to force the systems $\hat{H}_{k\beta}$ and $N_{k\beta}$ to have certain properties which will be essential in the derivation of the stability results. These restrictions on k(t) and β(t) are stated in the following definition.

Definition:

A pair of functions [k(t), β(t)] *is said to belong to the class* C, *viz* (k, β) ∈ C, *if the resulting systems* $\hat{H}_{k\beta}$ *and* $N_{k\beta}$ *satisfy respectively the conditions* C_H *and* C_N *stated below:*

C_H : *There exists an impulse response function in* $h_{k\beta}(t, z)$ *for the system* $\hat{H}_{k\beta}$ *such that*

(i) *The input-output relationship for the system* $\hat{H}_{k\beta}$ *is a linear mapping* $g(t) = \hat{H}_{k\beta} w$ *which can be expressed as*

$$g(t) = H_{k\beta}w = \hat{H}_{k\beta}(w)^* = \sum_{z\in T(t_0,t)} w(z)h_{k\beta}(t,z) \tag{17}$$

(ii)

$$\lim_{t\to\infty} \sum_{z\in T(t_0,T_1)} |h_{k\beta}(t,z)| = 0, \qquad any\ T_1 \geq t_0 \tag{18}$$

(iii) *The function* $A_{Tk\beta}(t)$, *defined by*

$$A_{Tk\beta}(t) \equiv \sum_{z\in T(t_0,T_1)} |h_{k\beta}(t,z)| \tag{19}$$

is a bounded function of t.

C_N : *The nonlinear relationship* $n_{k\beta}(x, t)$ *satisfies the following property*

$$\overline{n_{k\beta}}(\lambda) \equiv \sup_{\substack{x\in[-\lambda,\lambda] \\ t\in(-\infty,\infty)}} |n_{k\beta}(x,t)| < \infty, \qquad any\ \ \lambda \in [0,\infty] \tag{20}$$

3.2 Eventual bounds

The functions $E_1(t)$ and $E_2(t)$ are said to be eventual upper and lower bounds on the signal e(t) if, for every $\delta > 0$, one can find T_δ such that

$$E_2(t) - \delta \leq e(t) \leq E_1(t) + \delta, \quad \text{for all } t \geq T_\delta, \text{ any } \delta, \text{ some } T_\delta \tag{21}$$

Note that $\delta > 0$ may be taken as an arbitrarily small number. It can be readily shown that if $E_1(t)$ and $E_2(t)$ are eventual bounds on e(t), then $\overline{\lim_{t\to\infty}} E_1(t)$ and $\underline{\lim_{t\to\infty}} E_2(t)$ are also eventual bounds on e(t).

3.3 Definitions and Notations

The stability creiteria, to be presented in the following section, are stated in terms of a parameter $M_{Tk\beta}$ which is now defined. The symbol ε denotes an arbitrarily small positive number

$$X_{Tk\beta}(t, \varepsilon) = \{x: |x - r_k(t)| < A_{Tk\beta}(t)\,|n_{k\beta}(x, t)| + \varepsilon\} \tag{22}$$

$$m_{Tk\beta}(t, \varepsilon) = \sup_{x \in X_{Tk\beta}(t,\varepsilon)} |n_{k\beta}(x, t)| \tag{23}$$

$$M_{Tk\beta}(\varepsilon) = \overline{\lim_{t\to\infty}}\, m_{Tk\beta}(t, \varepsilon) \tag{24}$$

$$M_{Tk\beta} = \lim_{e\downarrow 0} M_{Tk\beta}(\varepsilon) \tag{25}$$

If $n_{k\beta}(x, t)$ is multi-valued at some $\hat{x}$, e.g., hysteresis, then $n_{k\beta}(\hat{x}, t)$ in (22) and (23) is understood to represent the largest value taken by $n_{k\beta}$ at $(\hat{x}, t)$.

Figure 4 shows a graphical representation of the set $X_{Tk\beta}(t, \varepsilon)$ and the number $m_{Tk\beta}(t, \varepsilon)$ as they relate to the graph of the non-linear characteristic $n_{k\beta}(x, t)$ for some value of t. The defintions in (22) to (25) can be combined in the following equivalent definition of $M_{Tk\beta}$:

$$M_{Tk\beta} = \lim_{\varepsilon \downarrow 0}\ \overline{\lim_{t\to\infty}} \left\{ \sup_{|x - r_k(t)| < A_{Tk\beta}(t)|n_{kb}(x,t)| + \varepsilon} |n_{k\beta}(x, t)| \right\} \tag{26}$$

For a given function $y(t)$, the notation $\bar{y}(a, b)$ will be used to represent the supremum over the interval $[a, b]$ of the absolute value of $y(t)$:

$$\bar{y}(a, b) \equiv \sup_{[a,b]} |y(t)|$$

The notation $y(t) \to 0$ will be used to imply that $y(t)$ goes to zero as t tends to infinity:

$$y(t) \to 0 \Leftrightarrow \lim_{t\to\infty} y(t) = 0$$

4. Main Results

The theorem which follows provide quantitative estimates for the eventual upper and lower bounds on the signal $e(t)$. Sufficient conditions for the boundedness, unboundedness and decay of $e(t)$ are then given in the subsequent corollaries. The stability behavior of the other output signals, namely $f(t)$, $u(t)$, and $g(t)$ are discussed in a subsequent section. The proof of the Theorem as well as the proofs of the five corollaries which follow are found in [4].

4.1 Theorem: Eventual bounds

If, for some $(k, \beta) \in C$, $M_{Tk\beta} < \infty$, *then*

$$E_1(t) \equiv r_k(t) + M_{Tk\beta}\, A_{Tk\beta}(t), \qquad E_2(t) \equiv r_k(t) - M_{Tk\beta}\, A_{Tk\beta}(t) \qquad (27)$$

are eventual bounds on e(t).

It should be emphasized that the eventual bounds given in (27) can provide qualitative as well as quantitative information about the behavior of the signal e(t) as $t \to \infty$. Recalling the definition of eventual bounds in § 3.2., it is easily seen that the functions $E_1(t)$ and $E_2(t)$ can be used to infer the boundedness, unboundedness and decay of e(t) follows from the corresponding behavior of the bounds $E_1(t)$ and $E_2(t)$. In the following corollaries we examine additional conditions that would guarantee the boundedness decay, or unboundedness of e(t). These additional conditions relate specifically to the effective input $r_k(t)$ and the function $A_{Tk\beta}(t)$.

4.2 Corollary 1: Boundedness

If, for some $(k, \beta) \in C$, $M_{Tk\beta} < \infty$ *and* $r_k(t)$ *is bounded, then* e(t) *is bounded.*

The validity of this corollary follows immediately from the theorem: since the condition $(k, \beta) \in C$ implies that $A_{Tk\beta}(t)$ is bounded, it follows that $E_1(t)$ and $E_2(t)$ are bounded, hence e(t) is bounded. It should be noted that the boundedness of $r_k(t)$ stipulated in Corollary 1 does not necessarily imply the the boundedness of r_1 or r_2. This can be seen from the expression of r_k in terms of r_1 and r_2 given in (14). This means that the output signal e(t) may be bounded even though r_1 and/or r_2 are unbounded. On the other hand, it can be shown that if r_1 and r_2 are bounded, then r_k is bounded, and Corollary 2 follows.

4.3 Corollary 2: Bounded-input, bounded-output

If, for some $(k, \beta) \in C$, $M_{Tk\beta} < \infty$, *then* e(t) *is bounded whenever* $r_1(t)$ *and* $r_2(t)$ *are bounded.*

4.4 Corollary 3: Decay

Let $(k, \beta) \in C$ *such that* $M_{Tk\beta} < \infty$ *and* $r_k(t) \to 0$. *Then* $e(t) \to 0$ *if either* $M_{Tk\beta} = 0$ *or* $A_{Tk\beta}(t) \to 0$.

The validity of this proposition follows from the theorem: the conditions of the corollary imply that the bounds $E_1, E_2 \to 0$, hence $e(t) \to 0$. Again, the condition $r_k(t) \to 0$ does not necessarily require that $r_1 \to 0$ or $r_2 \to 0$. On the other hand, it can be shown that if $r_1, r_2 \to 0$, then $r_k \to 0$, and Corollary 4 follows.

4.5 Corollary 4: Decaying-input, decaying-output

Let $(k, \beta) \in C$ *such that* $M_{Tk\beta} < \infty$. *Then* $e(t) \rightarrow 0$ *whenever* $r_1(t)$, $r_2(t) \rightarrow 0$ *if either* $M_{Tk\beta} = 0$ or $A_{Tk\beta}(t) \rightarrow 0$.

4.6 Corollary 5: Unboundedness

If, for some $(k, \beta) \in C$, $M_{Tk\beta} < \infty$ *and* $r_k(t)$ *is unbounded, then* $e(t)$ *is unbounded.*

The hypothesis of Corollary 5 implies that the bounds $E_1(t)$ and $E_2(t)$ are both unbounded, while their difference $E_1(t) - E_2(t) = 2\, M_{Tk\beta}\, A_{Tk\beta}(t)$ remains bounded, hence $e(t)$ is unbounded (see defintion of eventual bounds in (21)).

Remark: The condition $M_{Tk\beta} < \infty$, which appears in all of the preceeding results, lends itself to a simple graphical interpretation. Recalling from (24) and (25) that $M_{Tk\beta} = \lim_{\varepsilon \downarrow 0} \overline{\lim_{t \to \infty}}\, m_{Tk\beta}(t, \varepsilon)$, and referringto Figure 4 which depicts $m_{Tk\beta}(t, \varepsilon)$ for some t and ε , one may visualize $M_{Tk\beta}$ as the limiting value of $m_{Tk\beta}(t, \varepsilon)$ for small values of ε large values of t .

4.7 The condition $(k, \beta) \in C$

In condition C_H(iii), the stipulation that $A_{Tk\beta}$ be bounded is equivalent to the requirement that $\hat{H}_{k\beta}$ be 'bounded-input-bounded-output stable', i.e., if w(t) is bounded then g(t) is bounded, and conversely (Zadeh and Desoer 1963). This does not necessarily restrict the given system H to be bounded-input-bounded-output stable.

The condition C_N, as expressed in (20) , implies that the non-linear characteristic $n_{k\beta}(x, t)$ should be a bounded function of t for any finite value of x. This condition need not be satisfied by the given non-linear characteristic n(x, t). To illustrate, consider the example.

$$n(x, t) = x^2 \sin t + tx, \qquad (k, \ ) = (t, 1), \qquad n_{k\beta}(x, t) = x^2 \sin t$$

Note that $n_{k\beta}$ satisfies (20) even though n(x, t) is an unbounded function of t for every fixed value of x.

4.8 The signals f(t), u(t) and g(t)

Having established the boundedness, unboundedness, or decay of e(t), the stability behavior of the other output signals may be easily inferred from the relationship of these signals to e(t) and to the input signals r_1 and r_2 :

$$f(t) = n\,(e(t), t) \tag{28}$$

$$u(t) = r_2(t) + f(t) = r_2 + n\,(e(t), t) \tag{29}$$

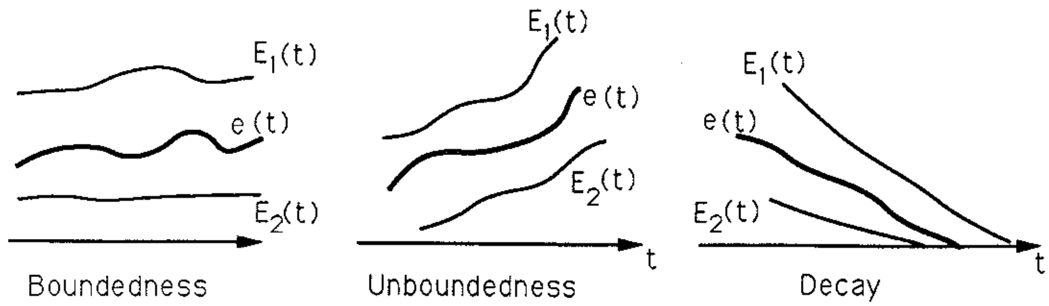

Figure 5. Relationship of $e(t)$, $E_1(t)$ and $E_2(t)$ under three types of dynamic response

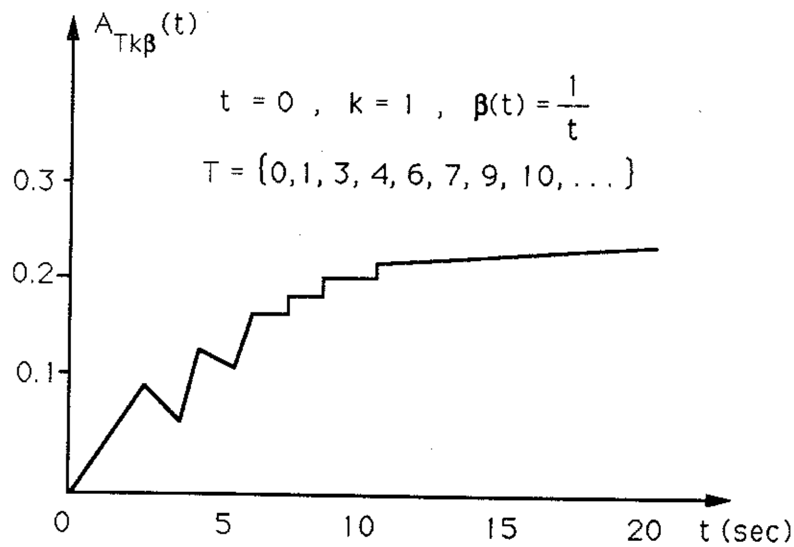

Figure 6. The function $A_{Tk\beta}(t)$ of the example

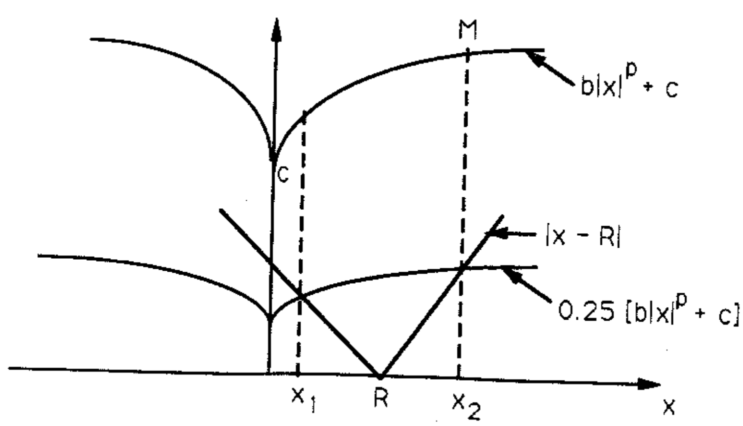

Figure 7. Graphical evaluation of M for the case $p < 1$ of the example

$$g(t) = r_1(t) - e(t) \tag{30}$$

Having verified that, say, $e(t)$ is bounded, the boundedness of $f(t)$ may be deduced from (28). If in addition $r_2(t)$ is known to be bounded, then $u(t)$ is bounded. Futhermore, if $r_1(t)$ is bounded, then (30) would imply that g is bounded. It should be emphasized that any such additional conditions that might be needed to establish the boundedness of $f(t)$, $u(t)$ or $g(t)$ are not necessary for the boundedness of $e(t)$. It is possible for $e(t)$ to be bounded when other signals in the system are unbounded.

5. Example

Let H be the characterized by the differential equation

$$t^2 \ddot{g} + 4t\,\dot{g} + (t^2 + 1)g = u, \qquad t \geq t_0 > 0 \tag{31}$$

The nonlinear characteristic $y = n(x, t)$ is not specified, but is assumed to satisfy the following sector restriction:

$$|n(x, t)| < (a + bt)\,|x|^{p(t)} + ct + d, \quad a, b, c, d,\ p(t) \geq 0, \quad \text{for all } t \geq t_0 \geq 0 \tag{32}$$

where $p(t)$ is any bounded function satisfying the condition

$$\overline{\lim_{t \to \infty}}\, p(t) = P \leq 1 \tag{33}$$

Let $(k, \beta) = (1, \frac{1}{t})$ for all $t \geq t_0 > 0$. The digital computer was conveniently used to determined the impulse response $h(t, z)$ of the system $\hat{H}_{k\beta}$ with skip sampling having $T = \{0, 1, 3, 4, 6, 7, 9, 10, \ldots\}$. Figure 6 shows the obtained plot of $A_{Tk\beta}(t)$, from which one reads

$$\overline{\lim_{t \to \infty}}\, A_{Tk\beta}(t) < 0.25 \tag{34}$$

The inputs $r_1(t)$ and $r_2(t)$ are not specified, but it is assumed that the resulting effective input $r_k(t)$ is bounded, with R defined as its limit superior:

$$\overline{\lim_{t \to \infty}}\, r_k(t) \equiv R < \infty \tag{35}$$

To verify condition C_N, note that (8) and (32) imply

$$n_{k\beta}(x, t) = \left| \frac{n(x,t) - x}{t} \right| \leq \frac{a + bt}{t} |x|^{p(t)} + c + \frac{d}{t} + \frac{|x|}{t} \tag{36}$$

Which, together with (33), imply that (20) is satisfied.

We shall now use corollaries 1 and 3 to determine conditions on the sector parameters a, b, c, and d, under which the boundedness or decay of e(t) can be guaranteed. For that, we need to examine $M_{Tk\beta}$. From (26) and (36), one has:

$$M_{Tk\beta} \leq \lim_{\varepsilon \downarrow 0}\ \overline{\lim_{t \to o}} \left\{ \begin{matrix} \sup \left[\frac{a + bt}{t} |x|^{p(t)} + c + \frac{d}{t} + \frac{x}{t} \right] \\ |x - r_k(t)| < A_{Tk\beta}(t) |n_{k\beta}(x,t)| + \varepsilon \end{matrix} \right\}$$

Using the expressions in (33), (34), and (35), one obtains:

$$M_{Tk\beta} \le M = \lim_{\varepsilon \downarrow 0} \begin{Bmatrix} \sup\left[b|x|^{P} + c\right] \\ |x - R| < 0.25\left[b|x|^{P} + c\right] + \varepsilon \end{Bmatrix} \qquad (37)$$

Figure 7 shows a graphical evaluation of M for the case P<1. The interval (x_1, x_2) is the set on the x-axis over which the supremum of $b|x|^P + c$ is evaluated. Evidently

$$M = b|x_2|^P + c, \quad P < 1. \qquad (38)$$

Thus, for P <1, the set (x_1, x_2) is bounded and $M_{Tk\beta} \le M < \infty$. Hence, by corollary 1, e(t) is bounded for any choice of the sector parameters a,b,c, and d. On the other hand, for P = 1, the set (x_1, x_2) is bounded if $0.25\ b < 1$. Hence for P = 1, e (t) is bounded for $b < \frac{1}{0.25} = 4$ and arbitrary values of a,c, and d. In either case, the bounds on e(t) may be expressed as

$$E_1(t) = r_k(t) + MA_{Tk\beta}(t), \qquad E_2 = r_k(t) - MA_{Tk\beta}(t)$$

where $A_{Tk\beta}$ (t) is the function plotted in Figure 6.

Next, we use Corollary 3 to obtain conditions for the decay of e(t). First we require the effective input $r_k(t)$ to be decaying. Furthermore, choosing b = c = 0, it follows from (37) that $M_{Tk\beta}$=M=0. Thus, $e(t) \to 0$ for any nonlinearity that satisfies the sector restriction

$$|n(x,t)| \le ax^{P(t)} + d \qquad (39)$$

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