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Algorithms for the Combined $H^2/H^\infty$
Model Reduction Problem

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Abstract

The problem of finding a reduced order model, optimal in the $H^2$ sense, to a given system model is a fundamental one in control system analysis and design. The addition of a $H^\infty$ constraint to the $H^2$ optimal model reduction problem results in a more practical yet computationally more difficult problem. Without the global convergence of probability-one homotopy methods the combined $H^2/H^\infty$ model reduction problem is difficult to solve. Several approaches based on homotopy methods have been proposed. The issues are the number of degrees of freedom, the well posedness of the finite dimensional optimization problem, and the numerical robustness of the resulting homotopy algorithm. Homotopy algorithms based on several formulations — input normal form; Ly, Bryson, and Cannon’s $2\times2$ block parametrization; a new nonminimal parametrization — are developed and compared here.

Keywords: reduced order model problem, $H^\infty$ control, $H^2$ control, probability-one homotopy algorithm.

AMS subject classifications: 93B20, 93B25, 93B40, 65H10.

1 Introduction

In a feedback control setting, order reduction techniques may be used either to simplify the plant for control design or to simplify the controller for ease of implementation. In either case, the resulting reduced-order systems must be constructed with their closed loop role in mind. Although numerous order reduction techniques have been proposed, it is clear from small-gain type arguments that the order reduction procedure should be to approximate the system frequency response to the greatest extent possible.

Several order reduction techniques have been proposed for approximating the frequency response of a given system. For example, frequency

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weighting has been studied in [5] in conjunction with balancing [12]. Moreover, Hankel norm reduction has been shown to have fundamental ramifications for frequency domain approximation [1], [2], [7]. An overview and discussion of these ideas is given in [3].

In the present paper we follow the approach of [8], which is based upon a state space $H^\infty$ formulation. In particular, by using a Riccati equation to enforce an $H^\infty$ constraint on the norm of the reduction error in conjunction with an $H^2$ upper bound or entropy cost [13], it was shown in [8] that $H^\infty$ constrained reduced order systems can be characterized by necessary conditions for optimality of the $H^2$ upper bound. The resulting algebraic conditions, which are a generalization of the "pure" $H^2$ optimality conditions given in [9], consist of nonstandard coupled Riccati and Lyapunov type matrix equations.

The purpose of the present paper is to make significant progress in developing novel, stable, globally convergent numerical algorithms for solving the optimality conditions for $H^2/H^\infty$ order reduction given in [8]. The approach we take is based on the construction of probability-one homotopy maps, similar to those developed for the $H^2$ order reduction problem in [6].

2 Statement of the Problem

Given the controllable and observable, time invariant, continuous time system
\begin{align}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t),
\end{align}
where $t \in [0, \infty)$, $A \in \mathbb{R}^{n \times n}$ is asymptotically stable, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, $D \in \mathbb{R}^{m \times p}$ ($m \leq p$) and the input $Du(t)$ is white noise with symmetric and positive definite intensity $V = DD^T$, find a $n_m$-th order model ($n_m < n$)
\begin{align}
\dot{x}_m(t) &= A_m x_m(t) + B_m Du(t), \\
y_m(t) &= C_m x_m(t),
\end{align}
where $A_m \in \mathbb{R}^{n_m \times n_m}$, $B_m \in \mathbb{R}^{n_m \times m}$, $C_m \in \mathbb{R}^{l \times n_m}$, which satisfies the following criteria:
(i) $A_m$ is asymptotically stable;
(ii) the transfer function of the reduced order model lies within $\gamma$ of the transfer function of the full order model in the $H^\infty$ norm, i.e.,
\begin{align}
\|H(s) - H_m(s)\|_\infty \leq \gamma
\end{align}
where

\[ H(s) \equiv EC(sI_n - A)^{-1}BD, \quad H_m(s) \equiv EC_m(sI_m - A_m)^{-1}B_mD, \]

\( \gamma > 0 \) is a given constant, \( E \in \mathbb{R}^{q \times l} \) \( q \geq l \) is a given constant matrix; and (iii) the \( H^2 \) model reduction criterion

\[ J(A_m, B_m, C_m) \equiv \lim_{t \to \infty} \mathcal{E} \left[ (y - y_m)^T R (y - y_m) \right] \]

(4)

is minimized, where \( \mathcal{E} \) is the expected value and \( R = E^T E \) is a symmetric and positive definite weighting matrix.

3 The auxiliary minimization problem

Define

\[ \check{n} \equiv n + n_m, \quad \hat{E} \equiv E\check{C}, \quad \hat{D} \equiv \bar{B} D, \]

\[ \check{A} \equiv \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}, \quad \check{B} \equiv \begin{pmatrix} B \\ B_m \end{pmatrix}, \quad \check{C} \equiv (C \quad -C_m), \]

(5)

\[ \check{R} \equiv \hat{E}^T \hat{E} = \check{C}^T R \check{C} = \begin{pmatrix} C^T R C & -C^T R C_m \\ -C_m^T R C & C_m^T R C_m \end{pmatrix}, \]

\[ \check{V} \equiv \check{D} \check{D}^T = \check{B} V \check{B}^T = \begin{pmatrix} B V B^T & B V B_m^T \\ B_m V B^T & B_m V B_m^T \end{pmatrix}. \]

(6)

The full order system (1) and the reduced order system (2) can be written as a single augmented system

\[ \dot{\check{x}}(t) = \check{A} \check{x}(t) + \check{D} u(t), \]

\[ \check{y}(t) = \check{C} \check{x}(t). \]

(7)

Using this notation the cost \( J(A_m, B_m, C_m) \) can be written as

\[ J(A_m, B_m, C_m) = \lim_{t \to \infty} \mathcal{E} \left[ (y - y_m)^T R (y - y_m) \right] = \lim_{t \to \infty} \mathcal{E}(\check{y}^T \check{R} \check{y}) = \lim_{t \to \infty} \mathcal{E}(\check{x}^T \check{R} \check{x}) = \text{tr} \left( \check{Q} \check{R} \right), \]

(8)

where \( \check{Q} \) satisfies

\[ \check{A} \check{Q} + \check{Q} \check{A}^T + \check{V} = 0. \]

(9)
**Lemma 1** [8]. Let $(A_m, B_m, C_m)$ be given and assume there exists $Q \in \mathbb{R}^{n \times n}$ satisfying

$$Q$$ is symmetric and nonnegative definite \hspace{1cm} (10)

and

$$\tilde{A} Q + Q \tilde{A}^T + \gamma^{-2} Q \tilde{R} Q + \tilde{V} = 0.$$

(11)

Then

$$(\tilde{A}, \tilde{D})$$ is stabilizable \hspace{1cm} (12)

if and only if

$A_m$ is asymptotically stable.

Furthermore, if (12) holds, then

$$\|H(s) - H_m(s)\|_\infty \leq \gamma,$$

(13)

$$\tilde{Q} \leq Q$$ \hspace{1cm} ($Q - \tilde{Q}$ is nonnegative definite),

and

$$\text{tr} \tilde{Q} \tilde{R} \equiv J(A_m, B_m, C_m) \leq J(A_m, B_m, C_m) \equiv \text{tr} Q \tilde{R}.$$ 

Hence the $H^\infty$ constraint is automatically enforced when a nonnegative definite solution to (11) is known to exist. Furthermore, the solution $Q$ provides an upper bound for the actual state covariance $\tilde{Q}$ along with a bound on the $H^2$ model reduction.

The satisfaction of (10)–(12) leads to (i) $A_m$ stable; (ii) a bound on the $H^\infty$ distance between the full order and reduced order systems; and (iii) an upper bound for the $H^2$ model-reduction criterion. The auxiliary minimization problem is to determine $(A_m, B_m, C_m)$ that minimizes $J(A_m, B_m, C_m)$ and thus provides a bound for the actual $H^2$ criterion $J(A_m, B_m, C_m)$.

$(A_m, B_m, C_m)$ is restricted to the set

$$S \equiv \{(A_m, B_m, C_m): \tilde{A} + \gamma^{-2} Q \tilde{R} \text{ is asymptotically stable,}$$

$$Q \text{ is symmetric positive definite,}$$

$$\text{and } (A_m, B_m, C_m) \text{ is controllable and observable } \}.$$ 

**4 A homotopy approach based on the input normal form**

**Theorem 1** [10]. Suppose $\tilde{A}_m$ is asymptotically stable. Then for every minimal $(\tilde{A}_m, \tilde{B}_m, C_m)$, i.e., $(\tilde{A}_m, \tilde{B}_m)$ is controllable and $(\tilde{A}_m, C_m)$ is
observable, there exist a similarity transformation $U$ and a positive definite matrix $\Omega = \text{diag}(\omega_1, \ldots, \omega_m)$ such that $A_m = U^{-1} \tilde{A}_m U$, $B_m = U^{-1} \tilde{B}_m$, and $C_m = \tilde{C}_m U$ satisfy

$$0 = A_m + A_m^T + B_m V B_m^T,$$
$$0 = A_m^T \Omega + \Omega A_m + C_m^T R C_m. \quad (14)$$

In addition,

$$\begin{align*}
(A_m)_{ii} &= -\frac{1}{2} (B_m V B_m^T)_{ii}, \\
\omega_i &= \frac{(C_m^T R C_m)_{ii}}{(B_m V B_m^T)_{ii}}, \\
(A_m)_{ij} &= \frac{(C_m^T R C_m)_{ij} - \omega_j (B_m V B_m^T)_{ij}}{\omega_j - \omega_i}, \quad \text{if } \omega_i \neq \omega_j.
\end{align*} \quad (15)$$

**Definition 1.** The triple $(A_m, B_m, C_m)$ satisfying (14) or (15) is said to be in input normal form.

To optimize $\mathcal{J}(A_m, B_m, C_m)$ over the open set $S$ under the constraints that symmetric positive definite $Q$ satisfies (11), and $(A_m, B_m, C_m)$ is in input normal form, the following Lagrangian is formed:

$$\mathcal{L}(A_m, B_m, C_m, \Omega, Q, \mathcal{P}, M_c, M_o)$$
$$\equiv \text{tr} [Q \tilde{R} + (\tilde{A} Q + Q \tilde{A}^T + \gamma^{-2} Q \tilde{R} Q + \tilde{V}) \mathcal{P}$$
$$+ (A_m + A_m^T + B_m V B_m^T) M_c + (A_m^T \Omega + \Omega A_m + C_m^T R C_m) M_o],$$

where the symmetric matrices $M_c, M_o,$ and $\mathcal{P} \in \mathbb{R}^{n \times n}$ are Lagrange multipliers. $\Omega = \text{diag}(\omega_1, \ldots, \omega_m)$ is related to the input normal form constraint. Setting $\partial \mathcal{L} / \partial Q = 0$ yields

$$0 = (\tilde{A} + \gamma^{-2} Q \tilde{R})^T \mathcal{P} + \mathcal{P} (\tilde{A} + \gamma^{-2} Q \tilde{R}) + \tilde{R}. \quad (16)$$

Partition $Q, \mathcal{P} \in \mathbb{R}^{n \times n}$ into

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} \mathcal{P}_1 & \mathcal{P}_{12} \\ \mathcal{P}_{12}^T & \mathcal{P}_2 \end{pmatrix} \quad (17)$$

where $Q_1, \mathcal{P}_1 \in \mathbb{R}^{n \times n}$ and $Q_2, \mathcal{P}_2 \in \mathbb{R}^{n \times n}$. Define

$$\mathcal{P} Q \equiv Z = \begin{pmatrix} Z_1 & Z_{12} \\ Z_{12}^T & Z_2 \end{pmatrix} \quad (18)$$
where
\[ Z_1 \equiv \mathcal{P}_1 Q_1 + \mathcal{P}_{12} Q_{12}^T, \quad Z_{12} \equiv \mathcal{P}_1 Q_{12} + \mathcal{P}_{12} Q_2, \]
\[ Z_{21} \equiv \mathcal{P}_{12}^T Q_1 + \mathcal{P}_2 Q_{12}^T, \quad Z_2 \equiv \mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2. \]
\[ \frac{\partial L}{\partial \Omega} = 0 \text{ and } \frac{\partial L}{\partial A_m} = 0 \text{ yield } \]
\[ 0 = 2M_c + 2\Omega M_o + 2(\mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2), \quad 0 = (A_m M_o)_{ii}, \quad 1 \leq i \leq n_m. \]

A straightforward calculation shows
\[ \frac{\partial L}{\partial B_m} = 2(\mathcal{P}_{12}^T B V + \mathcal{P}_2 B_m V) + 2M_c B_m V, \]
\[ \frac{\partial L}{\partial C_m} = 2(R C_m Q_2 - R C_{12}) + 2R C_m M_o \]
\[ + \gamma^{-2} \left[ - R C (Z_1^T Q_{12} + Z_{21}^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2) \right. \]
\[ \left. + R C_m (Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2) \right]. \]

**Theorem 2 [4].** The matrices \( M_c \) and \( M_o \) in (19) satisfy
\[ M_c = -\left( \frac{1}{2} S + \Omega M_o \right), \]
\[ (M_o)_{ii} = -\frac{1}{(A_m)_{ii}} \sum_{j \neq i}^{n_m} (A_m)_{ij} (M_o)_{ji}, \quad (20) \]
\[ (M_o)_{ij} = \frac{(S)_{ij} - (S)_{ii}}{2(\omega_j - \omega_i)}, \quad \text{if } \omega_j \neq \omega_i, \]

where
\[ S = 2(\mathcal{P}_{12}^T Q_{12} + \mathcal{P}_2 Q_2). \]

A homotopy approach based on the input normal form is now described. Let \( A_f, B_f, C_f, R_f, V_f, \) and \( \gamma_f \) denote \( A, B, C, R, V, \) and \( \gamma \) in the above and define
\[ A(\lambda) = A_0 + \lambda (A_f - A_0), \quad R(\lambda) = R_0 + \lambda (R_f - R_0), \]
\[ B(\lambda) = B_0 + \lambda (B_f - B_0), \quad V(\lambda) = V_0 + \lambda (V_f - V_0), \]
\[ C(\lambda) = C_0 + \lambda (C_f - C_0), \quad \gamma(\lambda) = \gamma_0 + \lambda (\gamma_f - \gamma_0). \]
For brevity, $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $R(\lambda)$, $V(\lambda)$, and $\gamma(\lambda)$ will be denoted by $A$, $B$, $C$, $R$, $V$, and $\gamma$ respectively in the following. Let

\[
H_{Bm}(\theta, \lambda) = \frac{\partial L}{\partial B_m} = 2(P_{12}^TB + P_2B_m)V + 2M_cB_mV,
\]

\[
H_{Cm}(\theta, \lambda) = \frac{\partial L}{\partial C_m} = 2R(C_mQ_2 - CQ_{12}) + 2RC_mM_o
\]

\[
+ \gamma^{-2}[-RC(Z_1^TQ_{12} + Z_{21}^TQ_2 + Q_1Z_{12} + Q_{12}Z_2)
\]

\[
+ RC_m(Q_{12}^TZ_{12} + Z_{12}^TQ_{12} + Q_2Z_2 + Z_2^TQ_2)],
\]

where

\[
\theta = \begin{pmatrix}
\text{Vec}(B_m) \\
\text{Vec}(C_m)
\end{pmatrix}
\]

denotes the independent variables $B_m$ and $C_m$, $M_o$ and $M_c$ satisfy (20), and $Q$ and $P$ satisfy respectively (11) and (16) with partitioned forms (17). Vec($P$) for a matrix $P \in \mathbb{R}^{p \times q}$ is the concatenation of its columns:

\[
\text{Vec}(P) = \begin{pmatrix}
P_1 \\
P_2 \\
\vdots \\
P_q
\end{pmatrix} \in \mathbb{R}^{pq}.
\]

The homotopy map is defined as

\[
\rho(\theta, \lambda) = \begin{pmatrix}
\text{Vec}[H_{Bm}(\theta, \lambda)] \\
\text{Vec}[H_{Cm}(\theta, \lambda)]
\end{pmatrix},
\]

and its Jacobian matrix is

\[
D\rho(\theta, \lambda) = (D_\theta\rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).
\]

Define

\[
\dot{H}_{Bm}(P^{(j)}, M_c^{(j)}) = 2(P_{12}^{T(j)}B + P_2^{T(j)}B_m)V + 2M_c^{(j)}B_mV,
\]

\[
\dot{H}_{Cm}(Q^{(j)}, Z^{(j)}, M_o^{(j)}) = 2R(C_mQ_2^{(j)} - CQ_{12}^{(j)}) + 2RC_mM_o^{(j)}
\]

\[
- \gamma^{-2}RC(Z_1^TQ_{12}^{(j)} + Z_{21}^TQ_2^{(j)} + Q_1Z_{12}^{(j)} + Q_{12}Z_2^{(j)}
\]

\[
+ Q_1^{(j)}Z_{12}^{(j)} + Q_1Z_{12}^{(j)} + Q_2Z_2^{(j)} + Q_{12}Z_2^{(j)})
\]

\[
+ \gamma^{-2}RC_m(Z_{12}^TQ_{12}^{(j)} + Z_{12}^TQ_{12}^{(j)} + Q_{12}^TZ_{12}^{(j)} + Q_{12}^TQ_{12}^{(j)}
\]

\[
+ Q_2^{(j)}Z_2^{(j)} + Z_2^TQ_2^{(j)})
\]

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where the superscript \((j)\) means \(\partial/\partial \theta_j\): \(Y^{(j)} = \partial Y/\partial \theta_j\). Using the above definitions, we have for \(\theta_j = (B_m)_{kl}\),

\[
\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} = \dot{H}_{B_m}(P^{(j)}, M_c^{(j)}) + 2(P_2 + M_c)E^{(k,l)}V,
\]

\[
\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} = \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}, M_o^{(j)}),
\]

and for \(\theta_j = (C_m)_{kl}\),

\[
\frac{\partial H_{B_m}}{\partial (C_m)_{kl}} = \dot{H}_{B_m}(P^{(j)}, M_c^{(j)}),
\]

\[
\frac{\partial H_{C_m}}{\partial (C_m)_{kl}} = \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}, M_o^{(j)}) + 2RE^{(k,l)}(Q_2 + M_o)
\]

\[
\quad + \gamma^{-2}RE^{(k,l)}(Z_{12}^TQ_{12} + Q_1^TQ_2 + Q_2^TQ_2 + Z_2^TQ_2),
\]

where \(E^{(k,l)}\) is a matrix of the appropriate dimension whose only nonzero element is \(e_{kl} = 1\). \(P^{(j)}\) and \(Q^{(j)}\) can be obtained by solving the Lyapunov equations

\[
0 = (\dot{A} + \gamma^{-2}\dot{Q}\dddot{R})Q^{(j)} + Q^{(j)}(\dot{A} + \gamma^{-2}Q\dddot{R})^T + \dddot{V}^{(j)}
\]

\[
+ \dot{A}^{(j)}Q + Q\dot{A}^{(j)}^T + \gamma^{-2}Q\dddot{R}^{(j)}Q,
\]

\[
0 = (\dot{A} + \gamma^{-2}\dot{Q}\dddot{R})^TP^{(j)} + P^{(j)}(\dot{A} + \gamma^{-2}Q\dddot{R}) + \dddot{R}^{(j)}
\]

\[
+ (\dot{A}^{(j)} + \gamma^{-2}Q^{(j)}\dddot{R} + \gamma^{-2}Q\dddot{R}^{(j)})^TP
\]

\[
+ P(\dot{A}^{(j)} + \gamma^{-2}Q^{(j)}\dddot{R} + \gamma^{-2}Q\dddot{R}^{(j)}).
\]

Similarly for \(\lambda\), using a dot to denote \(\partial/\partial \lambda\),

\[
\frac{\partial H_{B_m}}{\partial \lambda} = \dot{H}_{B_m}(\dot{P}, M_c) + 2P_{12}^T(\dot{B}V + B\dot{V}) + 2(P_2 + M_c)B_m\dot{V},
\]

\[
\frac{\partial H_{C_m}}{\partial \lambda} = \dot{H}_{C_m}(\dot{Q}, \dot{Z}, M_o) + 2\dddot{R}C_m(Q_2 + M_o) - 2(\dddot{R}C + R\dddot{C})Q_{12}
\]

\[
+ \gamma^{-2}\dot{R}h_\lambda - 2\gamma^{-3}\dot{\gamma}Rh_\lambda
\]

\[
- \gamma^{-2}R\dddot{C}(Z_1^TQ_{12} + Z_{21}^TQ_2 + Q_1Z_{12} + Q_{12}Z_2),
\]

where

\[
h_\lambda = -C(Z_1^TQ_{12} + Z_{21}^TQ_2 + Q_1Z_{12} + Q_{12}Z_2)
\]

\[
+ C_m(Q_1^TQ_{12} + Z_{12}^TQ_{12} + Q_2Z_2 + Z_2^TQ_2),
\]
and \( \dot{P} \) and \( \dot{Q} \) are obtained by solving the Lyapunov equations

\[
0 = (\dot{A} + \gamma^{-2} \dot{Q} \dot{R}) \dot{Q} + \dot{Q} (\dot{A} + \gamma^{-2} \dot{Q} \dot{R})^T + \dot{V} \\
+ \dot{A} Q + Q \dot{A}^T + \gamma^{-2} \dot{Q} \dot{R} Q - 2\gamma^{-3} \dot{\gamma} \dot{Q} \dot{R} Q,
\]

\[
0 = (\dot{A} + \gamma^{-2} \dot{Q} \dot{R})^T \dot{P} + \dot{P} (\dot{A} + \gamma^{-2} \dot{Q} \dot{R}) + \dot{R} \\
+ (\dot{A} + \gamma^{-2} \dot{Q} \dot{R} + \gamma^{-2} \dot{Q} \dot{R} - 2\gamma^{-3} \dot{\gamma} \dot{Q} \dot{R})^T \dot{P} \\
+ \dot{P} (\dot{A} + \gamma^{-2} \dot{Q} \dot{R} + \gamma^{-2} \dot{Q} \dot{R} - 2\gamma^{-3} \dot{\gamma} \dot{Q} \dot{R}).
\]

\[\tag{29}\]

5 Numerical algorithm for input normal form homotopy

The initial point \((\theta, \lambda) = (\theta_0, 0) = ((B_m)_0, (C_m)_0, 0)\) is ideally chosen so that the triple \(((A_m)_0, (B_m)_0, (C_m)_0)\) is in input normal form and satisfies \(\rho(\theta_0, 0) = 0\).

**Theorem 3** [12]. Suppose \(\bar{A}\) is asymptotically stable. Then for every minimal \((\bar{A}, \bar{B}, \bar{C})\), i.e., \((\bar{A}, \bar{B})\) is controllable and \((\bar{A}, \bar{C})\) is observable, there exist a similarity transformation \(T\) and a positive definite matrix \(\Lambda = \text{diag} (d_1, d_2, \ldots, d_n)\) with \(d_i \geq d_{i+1}\) such that \(A = T^{-1} \bar{A} T, B = T^{-1} \bar{B}, \) and \(C = \bar{C} T\) satisfy

\[
0 = A\Lambda + \Lambda A^T + BV B^T, \\
0 = A^T \Lambda + \Lambda A + C^T R C.
\]

**Definition 2.** The triple \((A, B, C)\) in the above theorem is balanced.

According to Moore [12], under certain conditions, the leading principal \(n_m \times n_m\) block of \(A\), the leading principal \(n_m \times m\) block of \(B\), and the leading principal \(l \times n_m\) block of \(C\) in balanced form are good approximations to the reduced order model. This suggests that the initial point \((\theta_0, 0)\) be chosen as follows:

1) Transform the given triple \((A_f, B_f, C_f)\) to balanced form \((A_b, B_b, C_b)\).
2) Partition \((A_b, B_b, C_b)\) as

\[
A_b = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}, \quad
B_b = \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix}, \quad
C_b = \begin{bmatrix}
C_1 & C_2
\end{bmatrix}.
\]

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3) \((A_0, B_0, C_0)\) is chosen as
\[
A_0 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} C_1 & 0 \end{pmatrix}.
\]

4) The initial point for the reduced order model is chosen as
\[
\bar{\theta}_0 = \begin{pmatrix} \text{Vec } (B_m)_0 \\ \text{Vec } (C_m)_0 \end{pmatrix} = \begin{pmatrix} \text{Vec } B_1 \\ \text{Vec } C_1 \end{pmatrix},
\]
and \((A_m)_0 = A_{11}\) by construction.

5) Transform the initial point \(((A_m)_0, (B_m)_0, (C_m)_0)\) to input normal form so that the initial reduced order model is
\[
((A_m)_0, (B_m)_0, (C_m)_0) = (T^{-1} (A_m)_0 T, \quad T^{-1} (B_m)_0, \quad (C_m)_0 T).
\]
The initial point for the homotopy map is then \((\theta_0, 0)\), where
\[
\theta_0 = \begin{pmatrix} \text{Vec } (B_m)_0 \\ \text{Vec } (C_m)_0 \end{pmatrix}.
\]

(In general, the truncation to obtain the approximate reduced order model should be based on the component costs instead of on the sizes of the balanced gains \(d_i\) as done above [14]. This explains why in some cases the above algorithm for choosing the initial points did not lead to a reduced order model with a minimal cost.)

The above method for choosing the initial point will not give a zero value for the homotopy at \(\lambda = 0\) unless the initial \(\gamma\) is chosen so that the term \(\gamma^{-2} \tilde{Q} \tilde{R}\) is negligible. The initial \(\gamma\) can be chosen as a sufficiently large positive number (\(\gamma(0) = \infty\) corresponds to \(\rho(\theta_0, 0) = 0\) exactly).

Once the initial point is chosen, the rest of the computation is as follows:
1) Set \(\lambda := 0, \theta := \theta_0\).
2) Calculate \(A_m\) from \(B_m\) and \(C_m\), \(\tilde{R}, \tilde{V}\), and compute \(Q\) and \(P\) according to (11) and (16).
3) Evaluate \(S\) from (21) and \(M_o\) and \(M_c\) according to (20).
4) Evaluate the homotopy map \(\rho(\theta, \lambda)\) in (23) and \(D\rho(\theta, \lambda)\) in (24).
5) Predict the next point \(Z^{(0)} = (\theta^{(0)}, \lambda^{(0)})\) on the homotopy zero curve using, e.g., a Hermite cubic interpolant.
6) For \(k := 0, 1, 2, \cdots\) until convergence do
\[
Z^{(k+1)} = [D\rho(Z^{(k)})]^{\dagger} \rho(Z^{(k)}).
\]
where \( [D\rho(Z)]^\dagger \) is the Moore-Penrose inverse of \( D\rho(Z) \). Let \( (\theta_1, \lambda_1) = \lim_{k \to \infty} Z^{(k)} \).
7) If \( \lambda_1 < 1 \), then set \( \theta := \theta_1, \lambda := \lambda_1 \), and go to step 2).
8) If \( \lambda_1 \geq 1 \), compute the solution \( \bar{\theta} \) at \( \lambda = 1 \). \( A_m \) is then obtained from \( B_m \) and \( C_m \).

An alternative strategy for choosing an initial point is as follows:
1) Modify \( A_f \) to \( A'_f = c_1 I + c_2 A_f \), where \( c_1 \leq 0 \) and \( c_2 \geq 0 \).
1) Transform \( (A'_f, B_f, C_f) \) to balanced form and choose \( (A'_0, B'_0, C'_0) \) as before.
3) Compute the initial reduced order model \( (\langle A_m \rangle_0, \langle B_m \rangle_0, \langle C_m \rangle_0) \) from the triple \( (A'_0, B'_0, C'_0) \) as before.

When \( c_1 = 0 \), \( c_2 = 1 \), this strategy reduces to the previous one. For some problems, our numerical experiments show that HOMPACK reaches \( \lambda > 1 \) in fewer steps with \( c_1 \neq 0 \) than with \( c_1 = 0 \). A modification to the homotopy map \( \rho(\theta, \lambda) \) in (23) is

\[
\rho_1(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0),
\]

where \( \theta_0 \) denotes the initial value of \( \theta \) at \( \lambda = 0 \). For some problems this homotopy map can be more efficient than the one in (23), while in other cases it can be less efficient.

6 Homotopy algorithm based on Ly’s formulation

Ly et al. [11] introduced another canonical form also with \( n_m m + n_m l \) parameters as in the input normal form formulation. The reduced order model is represented with respect to a basis such that \( A_m \) is a \( 2 \times 2 \) block-diagonal matrix (\( 2 \times 2 \) blocks with an additional \( 1 \times 1 \) block if \( n_m \) is odd) with \( 2 \times 2 \) blocks in the form

\[
\begin{pmatrix}
0 & 1 \\
* & *
\end{pmatrix},
\]

\( B_m \) is a full matrix, and

\[
C_m = \begin{pmatrix}
(C_m)_1 & (C_m)_2 & \cdots & (C_m)_r
\end{pmatrix}
\]

where

\[
(C_m)_i = \begin{pmatrix}
1 & * & \cdots & *
0 & * & \cdots & *
\end{pmatrix}^T.
\]
It is assumed that \((A_m, B_m, C_m)\) is in Ly’s form. Let \(\mathcal{I}\) be the set of indices of those elements of \(A_m\) which are parameters, i.e.,

\[
\mathcal{I} \equiv \{(2,1), (2,2), \ldots, (n_m, n_m)\}.
\]

To optimize \(\mathcal{J}(A_m, B_m, C_m)\) over the open set \(\mathcal{S}\) under the constraint that symmetric positive definite \(Q\) satisfies (11), and \((A_m, B_m, C_m)\) is in Ly’s form, the following Lagrangian is formed:

\[
\mathcal{L}(A_m, B_m, C_m, P, Q) \equiv \text{tr} \left[ Q \dot{R} + (\dot{A}Q + Q \dot{A}^T + \gamma^{-2} Q \dot{R}Q + \dot{V})P \right],
\]

where \(P \in \mathbb{R}^{n \times n}\) is a Lagrange multiplier. Setting \(\partial \mathcal{L}/\partial Q = 0\) yields (16). Partition \(Q, P \in \mathbb{R}^{n \times n}\) as in (17) and define \(PQ = Z\) as in (18). The partial derivatives of \(\mathcal{L}\) can be computed as

\[
\frac{\partial \mathcal{L}}{\partial (A_m)_{ij}} = 2 \left( P_{12}^T Q_{12} + P_{21} Q_{21} \right)_{ij}, \quad (i,j) \in \mathcal{I},
\]

\[
\frac{\partial \mathcal{L}}{\partial B_m} = 2(P_{12}^T B + P_{21} B_m V),
\]

\[
\frac{\partial \mathcal{L}}{\partial (C_m)_{ij}} = 2 \left( R C_m Q_{2} - R C Q_{12} \right)_{ij} + \gamma^{-2} \left[ -R C (Z_1^T Q_{12} + Z_2^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2) 
\right. \\
+ R C_m (Q_{12}^T Z_{12} + Z_1^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2) \bigg]_{ij}.
\]

Let \(A_f, B_f, C_f, R_f, V_f,\) and \(\gamma_f\) denote \(A, B, C, R, V,\) and \(\gamma\) in the above and define \(A(\lambda), B(\lambda), C(\lambda), R(\lambda), V(\lambda),\) and \(\gamma(\lambda)\) as in (22) and denote them by \(A, B, C, R, V,\) and \(\gamma\) respectively in the following. Let

\[
H_{A_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial A_m} = 2(P_{12}^T Q_{12} + P_{21} Q_{21}),
\]

\[
H_{B_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial B_m} = 2(P_{12}^T B + P_{21} B_m V),
\]

\[
H_{C_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial C_m} = 2R(C_m Q_2 - C Q_{12})
\]

\[
+ \gamma^{-2} \left[ -R C (Z_1^T Q_{12} + Z_2^T Q_2 + Q_1 Z_{12} + Q_{12} Z_2) 
\right. \\
+ R C_m (Q_{12}^T Z_{12} + Z_1^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2) \bigg],
\]

where in \(H_{A_m}\) only those elements corresponding to the parameter elements of \(A_m\) are of interest and

\[
\theta \equiv \begin{pmatrix} (A_m)_T \\ \text{Vec} (B_m) \\ \text{Vec} (C_m)_T \end{pmatrix}
\]

\[
(31)
\]
denotes the independent variables, \( Q \) and \( P \) satisfy respectively (11) and (16), \((A_m)_{I}\) is a vector consisting of those elements in \( A_m \) with indices in the set \( I \), i.e.,

\[
(A_m)_{I} = ((A_m)_{21}, (A_m)_{22}, \cdots, (A_m)_{nm})^T,
\]

and \((C_m)_{T}\) is the matrix obtained from rows \( T = \{2, \ldots, l\} \) of \( C_m \).

The homotopy map is defined as

\[
\rho(\theta, \lambda) = \begin{pmatrix}
\begin{bmatrix} H_{A_m}(\theta, \lambda) \end{bmatrix}_I \\
\text{Vec} \left[ H_{B_m}(\theta, \lambda) \right] \\
\text{Vec} \left[ H_{C_m}(\theta, \lambda) \right]_T
\end{pmatrix},
\]

(32)

and its Jacobian matrix is

\[
D\rho(\theta, \lambda) = (D_\theta \rho(\theta, \lambda), D_\lambda \rho(\theta, \lambda)).
\]

Define

\[
\begin{align*}
\dot{H}_{A_m}(P^{(j)}, Q^{(j)}) &= 2(P_{12}^{T(j)} Q_{12} + P_{12}^{T(j)} Q_{12}^{(j)} + P_{2}^{(j)} Q_{2} + P_{2} Q_{2}^{(j)}), \\
\dot{H}_{B_m}(P^{(j)}) &= 2(P_{12}^{T(j)} B + P_{2}^{(j)} B_m)V, \\
\dot{H}_{C_m}(Q^{(j)}, Z^{(j)}) &= 2R(C_m Q_{2}^{(j)} - C Q_{12}^{(j)}) \\
&\quad - \gamma^{-2} R C (Z_{12}^{T(j)} Q_{12} + Z_{21}^{T(j)} Q_{2} + Z_{1}^{T(j)} Q_{12} + Z_{21}^{T(j)} Q_{2}^{(j)}) \\
&\quad + Q_{1}^{(j)} Z_{12} + Q_{1} Z_{12}^{(j)} + Q_{12}^{(j)} Z_{2} + Q_{12} Z_{2}^{(j)} \\
&\quad + \gamma^{-2} R C_{m} (Z_{12}^{T(j)} Q_{12} + Z_{21}^{T(j)} Q_{12}^{(j)} + Q_{12}^{T(j)} Z_{12} + Q_{12} Z_{12}^{(j)}) \\
&\quad + Q_{2}^{(j)} Z_{2} + Z_{2}^{T(j)} Q_{2} + Q_{2} Z_{2}^{(j)} + Z_{2}^{T(j)} Q_{2}^{(j)}),
\end{align*}
\]

(33)

where the superscript \((j)\) means \( \partial/\partial \theta_j \). Using the above definitions, we have for \( \theta_j = (A_m)_{kl} \), where \((k, l) \in I\),

\[
\begin{align*}
\frac{\partial \dot{H}_{A_m}}{\partial (A_m)_{kl}} &= \dot{H}_{A_m}(P^{(j)}, Z^{(j)}), \\
\frac{\partial \dot{H}_{B_m}}{\partial (A_m)_{kl}} &= \dot{H}_{B_m}(P^{(j)}), \\
\frac{\partial \dot{H}_{C_m}}{\partial (A_m)_{kl}} &= \dot{H}_{C_m}(Q^{(j)}, Z^{(j)}),
\end{align*}
\]

(34)
for $\theta_j = (B_m)_{kl}$,

$$\frac{\partial H_{Am}}{\partial (B_m)_{kl}} = \dot{H}_{Am}(\mathcal{P}^{(j)}, Q^{(j)}),$$

$$\frac{\partial H_{Bm}}{\partial (B_m)_{kl}} = \dot{H}_{Bm}(\mathcal{P}^{(j)}) + 2\mathcal{P}_2 E^{(k,l)} V,$$

$$\frac{\partial H_{Cm}}{\partial (B_m)_{kl}} = \dot{H}_{Cm}(Q^{(j)}, Z^{(j)}),$$

(35)

and for $\theta_j = (C_m)_{kl}$, where $k > 1$,

$$\frac{\partial H_{Am}}{\partial (C_m)_{kl}} = \dot{H}_{Am}(\mathcal{P}^{(j)}, Q^{(j)}),$$

$$\frac{\partial H_{Bm}}{\partial (C_m)_{kl}} = \dot{H}_{Bm}(\mathcal{P}^{(j)}),$$

(36)

$$\frac{\partial H_{Cm}}{\partial (C_m)_{kl}} = \dot{H}_{Cm}(Q^{(j)}, Z^{(j)}) + 2\mathcal{P}_2 E^{(k,l)} Q_2 + 
\gamma^{-2}\mathcal{P}_2 E^{(k,l)} (Z_{12}^T Q_{12} + Q_{12}^T Z_{12} + Q_2^T Z_2 + Z_2^T Q_2),$$

where $\mathcal{P}^{(j)}$ and $Q^{(j)}$ can be obtained by solving the Lyapunov equation (27). Similarly for $\lambda$, using a dot to denote $\partial/\partial \lambda$,

$$\frac{\partial H_{Am}}{\partial \lambda} = \dot{H}_{Am}(\dot{\mathcal{P}}, \dot{Q}),$$

$$\frac{\partial H_{Bm}}{\partial \lambda} = \dot{H}_{Bm}(\dot{\mathcal{P}}) + 2\mathcal{P}_1^T (\dot{B} V + B \dot{V}) + 2\mathcal{P}_2 B_\lambda \dot{V},$$

(37)

$$\frac{\partial H_{Cm}}{\partial \lambda} = \dot{H}_{Cm} (\dot{Q}, \dot{Z}) - 2(\dot{R} C + R \dot{C}) Q_{12} + 2\dot{R} C_2 Q_2
+ \gamma^{-2}\mathcal{R} h_\lambda - 2\gamma^{-3}\mathcal{R} h_\lambda
- \gamma^{-2} R \dot{C} (Z_{1}^T Q_{12} + Z_{21}^T Q_2 + Q_1 Z_{12} + Q_1 Z_2),$$

where

$$h_\lambda = -C (Z_{1}^T Q_{12} + Z_{21}^T Q_2 + Q_1 Z_{12} + Q_1 Z_2)
+ C_m (Q_{12}^T Z_{12} + Z_{12}^T Q_{12} + Q_2 Z_2 + Z_2^T Q_2),$$

and $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are obtained by solving (29).

Choose the initial $\gamma$ so that $\gamma^{-2}$ is approximately zero. The initial point $(\theta, \lambda) = (\theta_0, 0)$ is chosen so that the triple $((A_m)_0, (B_m)_0, (C_m)_0)$ is in Ly' s form and satisfies $\rho(\theta_0, 0) = 0$. This can be done as follows:
1) Obtain the initial reduced order model \((A_m)_0, (B_m)_0, (C_m)_0\) in balanced form in the same way as for the input normal form approach.
2) Transform the balanced \((A_m)_0, (B_m)_0, (C_m)_0\) to Ly's form, and build \(\theta_0\) as described in (31).

The homotopy curve tracking computation is the same as described in Section 5.

7 Homotopy algorithm based on over-parametrization formulation

To optimize \(J(A_m, B_m, C_m)\) over the open set \(S\) under the constraint that symmetric positive definite \(Q\) satisfies (11), the following Lagrangian is formed:

\[
\mathcal{L}(A_m, B_m, C_m, P, Q) = \text{tr} \left[ Q\bar{R} + (\bar{A}Q + Q\bar{A}^T + \gamma^{-2}Q\bar{R}Q + \bar{V})P \right]
\]

where \(P \in \mathbb{R}^{n\times n}\) is a Lagrange multiplier. Setting \(\partial\mathcal{L}/\partial Q = 0\) yields (16). Partition \(Q, P \in \mathbb{R}^{n\times n}\) as in (17) and define \(PQ = Z\) as in (18). A straightforward calculation shows

\[
\frac{\partial \mathcal{L}}{\partial A_m} = 2(P_{12}^TQ_{12} + P_2Q_2),
\]

\[
\frac{\partial \mathcal{L}}{\partial B_m} = 2(P_{12}^TBV + P_2B_mV),
\]

\[
\frac{\partial \mathcal{L}}{\partial C_m} = 2(RC_mQ_2 - RCQ_{12})
\]

\[
+ \gamma^{-2}[-RC(Z_1^TQ_{12} + Z_2^TQ_2 + Q_1Z_{12} + Q_{12}Z_2)
\]

\[
+ RC_m(Q_2^TZ_{12} + Z_1^TQ_{12} + Q_2Z_2 + Z^T_2Q_2)].
\]

Let \(A_f, B_f, C_f, R_f, V_f\), and \(\gamma_f\) denote \(A, B, C, R, V\), and \(\gamma\) in the above and define \(A(\lambda), B(\lambda), C(\lambda), R(\lambda), V(\lambda)\), and \(\gamma(\lambda)\) as in (22) and denote them by \(A, B, C, R, V\), and \(\gamma\) respectively in the following. Let

\[
H_{A_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial A_m} = 2(P_{12}^TQ_{12} + P_2Q_2),
\]

\[
H_{B_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial B_m} = 2(P_{12}^TB + P_2B_m)V,
\]

\[
H_{C_m}(\theta, \lambda) = \frac{\partial \mathcal{L}}{\partial C_m} = 2R(C_mQ_2 - CQ_{12})
\]

\[
+ \gamma^{-2}[-RC(Z_1^TQ_{12} + Z_2^TQ_2 + Q_1Z_{12} + Q_{12}Z_2)
\]

\[
+ RC_m(Q_2^TZ_{12} + Z_1^TQ_{12} + Q_2Z_2 + Z^T_2Q_2)],
\]
where
\[
\theta \equiv \begin{pmatrix}
\text{Vec} \ (A_m) \\
\text{Vec} \ (B_m) \\
\text{Vec} \ (C_m)
\end{pmatrix}
\] (39)
denotes the independent variables, \(Q\) and \(P\) satisfy respectively (11) and (16).

Define
\[
\rho(\theta, \lambda) = \begin{pmatrix}
\text{Vec} \ [H_{A_m}(\theta, \lambda)] \\
\text{Vec} \ [H_{B_m}(\theta, \lambda)] \\
\text{Vec} \ [H_{C_m}(\theta, \lambda)]
\end{pmatrix},
\] (40)
whose Jacobian matrix is
\[
D\rho(\theta, \lambda) = (D_{\theta}\rho(\theta, \lambda), D_{\lambda}\rho(\theta, \lambda)).
\]

Note that \(\theta\) in (39) has \(n_m^2 + n_m m + n_m l\) components, more than the minimal number \(n_m m + n_m l\) of the input normal form and \(L_y\) formulations. Because of this over-parametrization, the Jacobian matrix of \(\rho\) is rank deficient. The homotopy map is thus defined as
\[
\hat{\rho}(\theta, \lambda) = \lambda \rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0),
\] (41)
which guarantees a well conditioned full rank Jacobian matrix along the whole path except at the solution corresponding to \(\lambda = 1\). The Jacobian matrix of \(\hat{\rho}\) is given by
\[
D\hat{\rho}(\theta, \lambda) = (\lambda D_{\theta}\rho(\theta, \lambda) + (1 - \lambda)I, \ \rho(\theta, \lambda) + \lambda D_{\lambda}\rho(\theta, \lambda) - (\theta - \theta_0)).
\] (42)

To find \(D_{\theta}\rho(\theta, \lambda)\), define \(\hat{H}_{A_m}(P^{(j)}, Q^{(j)}), \hat{H}_{B_m}(P^{(j)}), \text{and } \hat{H}_{C_m}(Q^{(j)}, Z^{(j)})\) as in (33). For \(\theta_j = (A_m)_{kl}\),
\[
\frac{\partial H_{A_m}}{\partial (A_m)_{kl}} = \hat{H}_{A_m}(P^{(j)}, Q^{(j)}),
\]
\[
\frac{\partial H_{B_m}}{\partial (A_m)_{kl}} = \hat{H}_{B_m}(P^{(j)}),
\]
\[
\frac{\partial H_{C_m}}{\partial (A_m)_{kl}} = \hat{H}_{C_m}(Q^{(j)}, Z^{(j)}),
\] (43)
for $\theta_j = (B_m)_{kl}$,

$$
\frac{\partial H_{A_m}}{\partial (B_m)_{kl}} = \hat{H}_{A_m}(p^{(j)}, Q^{(j)}),
$$

$$
\frac{\partial H_{B_m}}{\partial (B_m)_{kl}} = \hat{H}_{B_m}(p^{(j)}) + 2p_2E^{(k,l)}V,
$$

$$
\frac{\partial H_{C_m}}{\partial (B_m)_{kl}} = \hat{H}_{C_m}(Q^{(j)}, Z^{(j)}),
$$

(44)

and for $\theta_j = (C_m)_{kl}$,

$$
\frac{\partial H_{A_m}}{\partial (C_m)_{kl}} = \hat{H}_{A_m}(p^{(j)}, Q^{(j)}),
$$

$$
\frac{\partial H_{B_m}}{\partial (C_m)_{kl}} = \hat{H}_{B_m}(p^{(j)}),
$$

$$
\frac{\partial H_{C_m}}{\partial (C_m)_{kl}} = \hat{H}_{C_m}(Q^{(j)}, Z^{(j)}) + 2RE^{(k,l)}Q_2
$$

$$
+ \gamma^{-2}RE^{(k,l)}(Z_{12}^TQ_{12} + Q_{12}^TZ_{12} + Q_{2}^TZ_2 + Z_2^TQ_2),
$$

(45)

where $p^{(j)}$ and $Q^{(j)}$ can be obtained by solving the Lyapunov equation (27). The derivative of the homotopy map with respect to $\lambda$ is given by (37) and (42).

Choose the initial $\gamma$ so that $\gamma_0^{-2}$ is approximately zero. The initial point $(\theta, \lambda) = (\theta_0, 0)$ is chosen so that the triple $((A_m)_0,(B_m)_0,(C_m)_0)$ is in balanced form and satisfies $\rho(\theta_0, 0) = 0$. This can be done as follows:

1) Obtain the initial reduced order model $((A_m)_0,(B_m)_0,(C_m)_0)_b$ in balanced form in the same way as for the input normal form approach.

2) Build $\theta_0$ from $((A_m)_0,(B_m)_0,(C_m)_0)_b$ as described in (39).

The homotopy curve tracking computation is the same as described in Section 5.

8 Numerical Results

The following systems are solved by the homotopy algorithms discussed in the previous sections. The homotopy curve tracking was done with HOMPACK [16].

Systems with transfer functions of the form

$$
H(s) = \frac{(s - 1)^q}{(s + 1)^{10}},
$$

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where \( q = 0, \cdots, 4 \), are studied. For these systems, controller canonical form realizations are given by

\[
A = \begin{pmatrix}
-10 & -45 & -120 & -210 & -250 & -210 & -120 & -45 & -10 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
B = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T.
\]

For \( q = 0 \), \( C = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0) \).
For \( q = 1 \), \( C = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ -1 \ 1) \).
For \( q = 2 \), \( C = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -2 \ 1) \).
For \( q = 3 \), \( C = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -3 \ 3 \ -1) \).
For \( q = 4 \), \( C = (0 \ 0 \ 0 \ 0 \ 0 \ 1 \ -4 \ 6 \ -4 \ 1) \).

The \( H^\infty \) error \( \|H(s) - H_m(s)\|_\infty \) for the balanced reduced model of order 4 and the corresponding Enns-Glover bounds [5] [7] are:

\[
\begin{array}{c|c|c}
q & H^\infty \text{ error} & \text{Enns-Glover bound} \\
0 & 0.017251178 & 0.021955271 \\
1 & 0.031901448 & 0.042197266 \\
2 & 0.057214882 & 0.079880388 \\
3 & 0.098520472 & 0.14841709 \\
4 & 0.16125935 & 0.27001110 \\
\end{array}
\]

For \( n_m = 4 \) and \( q = 0 \), solutions of the auxiliary minimization problem are obtained for \( \gamma \geq 0.0178 \) using the input normal form approach. For \( \gamma < \gamma_{min} \equiv 0.0178 \), the Riccati equation solver fails and therefore no solution can be found. Let \( H_m(s) \) be the transfer function of the reduced order model obtained by minimizing \( \mathcal{J} \). In Fig. 1, \( \|H(s) - H_m(s)\|_\infty \) (solid line), 100 \( \mathcal{J} \) (dotted line), and 100 \( J \) (dashed line) are plotted against \( \gamma \). As shown in the figure, as \( \gamma \) decreases, \( \|H(s) - H_m(s)\|_\infty \) also decreases while both \( \mathcal{J} \) and \( J \) increase. As can be seen from the figure, \( \mathcal{J} \) is a close bound for \( J \) until \( \gamma \) becomes very small. To show the tradeoff between the \( H^2 \) cost and the \( H^\infty \) error \( \|H(s) - H_m(s)\|_\infty \), it is useful to plot \( \|H(s) - H_m(s)\|_\infty \) against
Fig. 1. $\|H(s) - H_m(s)\|_\infty$ (solid), $100\mathcal{J}(\text{dotted})$, and $100\mathcal{J}(\text{dashed})$ versus $\gamma$.

Fig. 2. $\|H(s) - H_m(s)\|_\infty$ versus $J$ for $q = 0$, balanced model at "x".
Fig. 3. $\|H(s) - H_m(s)\|_\infty$ versus $J$ for $q = 1$, balanced model at "x".

Fig. 4. Ratio of $H^\infty$ error at $\gamma = \gamma_{\text{min}}$ to that at $\gamma = \infty$ versus exponent $q$. 
Fig. 5. Bode plots of $H(s) - H_m(s)$ for $H^2/H^\infty$ (solid), $H^2$ (dashed), and balanced (dotted) models with $q = 0$.

Fig. 6. Poles ("\(\times\)"") and zeros ("\(\circ\)"") of the transfer function of the reduced order model for $q = 0$, $n_m = 4$ at $\gamma = 0.0178$. 

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$J$ (with $\gamma$ as the parameter of the curve), as shown in Fig. 2. In Fig. 2, the point marked by "x" corresponds to the balanced reduced model, which has both large $H^2$ cost and large $H^\infty$ error $\|H(s) - H_m(s)\|_\infty$, relative to the $H^2/H^\infty$ reduced order model. The ratio of $H^\infty$ error at $\gamma = \gamma_{\min}$ to that at $\gamma = \infty$ is 0.8071, which indicates that there is about 20% improvement of the reduced order model with $\gamma = \gamma_{\min}$ over the reduced order model without the $H^\infty$ constraint. The reduced order models of order 2, 3, 6 were also found, and have qualitative behavior similar to the $n_m = 4$ case.

For $q = 1, \cdots, 4$, the same calculations are carried out. Fig. 3 shows similar results to those in Fig. 2 for $q = 1$ with an improvement of about 19%. As $q$ increases, the improvement of the optimal reduced order model over the balanced reduced order model decreases. In Fig. 4, the ratio of $H^\infty$ error at $\gamma = \gamma_{\min}$ to that at $\gamma = \infty$ is plotted against $q$ for $q = 0, \cdots, 4$. The $H^\infty$ norm improvement of the optimal reduced order model with the $H^\infty$ constraint over that without the $H^\infty$ constraint is $1 - \text{ratio}$. As $q$ increases, the improvement decreases.

In Fig. 5, the Bode plots of $H(s) - H_m(s)$ for the system with $q = 0$, $n_m = 4$, and $\gamma = \gamma_{\min} = 0.0178$ are shown. The reduced order model with the $H^\infty$ constraint at $\gamma = \gamma_{\min}$ is shown by the solid line; the balanced reduced order model is shown by the dotted line; the reduced order model without the $H^\infty$ constraint is shown by the dashed line. The magnitude plots show that as $\gamma$ goes to $\gamma_{\min}$, the $H^\infty$ error becomes increasingly "all pass", that is, flat over a wide frequency range, which indicates $H^\infty$ optimality of the reduced order model. Fig. 6 shows the poles and zeros of the transfer function of the reduced order model for the system with $q = 0$ and $n_m = 4$ at $\gamma = \gamma_{\min}$.

As another example, consider the system defined by

$$A = \begin{pmatrix} -2 & -8 \\ 0 & -8 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad C = B^T.$$

It is easy to verify that the system is balanced and the singular values are all equal to 1, i.e.,

$$A + A^T + BB^T = 0,$$

$$A^T + A + C^TC = 0.$$

The $H^\infty$ error of the balanced reduced model of order 1 is 2 and the Enns-Glover bound is also 2. Optimal reduced models of order 1 are found by the input normal form homotopy approach for $\gamma \geq \gamma_{\min} = 1.011$. Fig. 7 shows the $H^\infty$ error versus the $H^2$ cost. The point $\times$ corresponds to
the balanced reduced order model. The ratio of the $H^\infty$ error at $\gamma = \gamma_{\min}$ to that at $\gamma = \infty$ is 0.6249, which indicates that there is about 37.5% improvement of the reduced order model with $\gamma = \gamma_{\min}$ over the reduced order model without the $H^\infty$ constraint, and a 50% improvement over the balanced reduced order model.

In Fig. 8, $50 \|H(s) - H_m(s)\|_\infty$ (solid line), $\mathcal{J}$ (dotted line), and $J$ (dashed line) are plotted against $\gamma$. Unlike the previous systems, even for small $\gamma$, the actual error $\|H(s) - H_m(s)\|_\infty$ is very close to its bound $\gamma$, and $\mathcal{J}$ is a very close bound for the $H^2$ error $J$. The Bode plots of $H(s) - H_m(s)$ are shown in Fig. 9, where the reduced order model with the $H^\infty$ constraint at $\gamma = \gamma_{\min}$ is shown by the solid line; the balanced reduced order model is shown by the dotted line; the reduced order model without the $H^\infty$ constraint is shown by the dashed line. Again the reduced order model for $\gamma = \gamma_{\min}$ indicates close to all pass model reduction error. The reduced order model transfer function at $\gamma = \gamma_{\min}$ has a single pole at $s = -129.1642$.

9 Conclusion

One of the main conclusions for this study is that the more degrees of freedom that a formulation uses, the more robust is the resulting numerical algorithm. Both the input normal form and Ly form homotopies are very efficient for both the $H^2$ optimal and the combined $H^2/H^\infty$ model reduction problems. However, they may fail to exist or be very ill conditioned [6]. The over-parametrization formulation solves the ill conditioning issue, but introduces singularity at the solution and may fail for a high dimensional system, which will inevitably have a high order singularity at the solution.

Solving the $H^2$ optimal model order reduction problem may be well worth the effort (compared to simple balancing), as shown by the last example above. The examples also proved the worth of adding the $H^\infty$ constraint, resulting in a difficult combined $H^2/H^\infty$ problem. Finally, globally convergent homotopy methods are a viable approach to the computationally very difficult combined $H^2/H^\infty$ model order reduction problem.

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Fig. 7. \( \|H(s) - H_m(s)\|_\infty \) versus \( J \).

Fig. 8. \( 50 \|H(s) - H_m(s)\|_\infty \) (solid), \( \mathcal{J} \) (dotted), and \( J \) (dashed) versus \( \gamma \).
Fig. 9. Bode plots of $H(s) - H_m(s)$ for $H^2/H^\infty$ (solid), $H^2$ (dashed), and balanced (dotted) reduced order model.

References


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