Hodiex: A Sixth Order Accurate Method for Solving Elliptic PDEs

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HODIEX: A HIGH ORDER ACCURATE METHOD FOR SOLVING GENERAL LINEAR ELLIPTIC PDEs *

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ABSTRACT

This paper describes a method for discretizing general linear two dimensional elliptic PDEs with variable coefficients, $Lu = g$, which achieves high orders of accuracy on an extended range of problems. The method can be viewed as an extension of the *ELLPACK*[6] discretization module *HODIE*[4] (an acronym derived from "High Order Difference Approximation with Identity Expansion"), which achieves high orders of accuracy on a more limited class of problems. We thus call this method *HODIEX*. *HODIE* will achieve $\mathcal{O}(h^4)$ accuracy on general problems with no cross derivatives and order as high as $\mathcal{O}(h^6)$ on certain problems with constant coefficients. *HODIEX* will achieve $\mathcal{O}(h^4)$ on general problems including those with cross derivatives and $\mathcal{O}(h^6)$ or higher on many other problems. The more smooth the solution and coefficients the higher order is possible up to a maximum theoretical value which is problem dependent. The values of the approximate solution U are determined at mesh points by solving the system $L_hU = I_hg$ where L_hU is a linear combination of values of U at the stencil points and I_hg is a linear combination of g at a set of evaluation points. An advantage of *HODIE* methods, including the one described here, is that they are based on a compact 9-point stencil which yields linear systems with a smaller bandwidth than if a larger stencil were used to achieve higher accuracy. Details on finding the discrete operators L_h and I_h and programming considerations are discussed. Performance on several test problems is reported, along with comparisons with *HODIE* and *9 point star* (a classical finite difference method which achieves $\mathcal{O}(h^2)$ accuracy).

1. INTRODUCTION

A general linear second order elliptic PDE in two dimensions with variable coefficients may be written

$$Lu \equiv au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g, \tag{1.1}$$

where a, b, c, d, e, f, and g are smooth functions of x and y on a simply connected domain D with a piecewise smooth boundary ∂D. Assume that grid size $h_x = h_y = h$ and n is the number of interior grid points in each dimension. In this paper we assume Dirichlet boundary conditions. We also assume that D is rectangular, although extensions to nonrectangular D are possible. Using a

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standard compact 9 point stencil, approximation of the various partial derivatives of u at a point (x_i, y_j) are obtained with traditional finite differences by the formulas

$$\begin{aligned} u_{xy} &\approx \frac{U_{i+1,j+1} - U_{i+1,j-1} - U_{i-1,j+1} + U_{i-1,j-1}}{4h^2}, \ u_{xx} \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}, \\ u_{yy} &\approx \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{h^2}, \ u_x \approx \frac{U_{i+1,j} - U_{i-1,j}}{2h}, \ u_y \approx \frac{U_{i,j+1} - U_{i,j-1}}{2h}, \end{aligned} \tag{1.2}$$

where $U_{i,j} = U(x_i, y_j)$ at the nine stencil points. These approximations on an $n \times n$ grid result in a linear system of n^2 equations of the form

$$AU = R \tag{1.3}$$

which, for sufficiently smooth problems, will have $\mathcal{O}(h^2)$ accuracy. Collatz[1] demonstrated that a more accurate approximation could be obtained by averaging the source term g over additional points. His *mehrstellenvarfahren* method (literally more points method) replaces derivatives of g with divided differences and determines coefficients by equating coefficients of linear combinations of Taylor expansions of u and g. Lynch and Rice[4] derive the *HODIE* method which use no derivatives and also achieves a higher order. Lynch's *HODIE* program was incorporated in *ELLPACK*[6], a large package for solving elliptic PDEs, of which *HODIE* is only one of some 50 different modules.

HODIE can achieve $\mathcal{O}(h^4)$ and $\mathcal{O}(h^6)$ accuracy, although $\mathcal{O}(h^6)$ accuracy is achieved only on a very limited class of problems. To achieve $\mathcal{O}(h^4)$ accuracy, *HODIE* requires no cross derivatives, or equivalently $b = 0$. If $b = 0$ and $a = c = 1$ then at least $\mathcal{O}(h^4)$ accuracy is achieved. Finally, to achieve $\mathcal{O}(h^6)$ accuracy requires $b = 0$, $a = c = 1$, $d_y = e_x$, f=constant, and no Neumann boundary conditions. In *HODIEX* we extend Lynch's method to allow higher orders on a more general class of problems and in fact achieve a minimum order of accuracy of $\mathcal{O}(h^4)$ on any general linear second order PDE as described by equation (1.1).

2. THE HODIEX APPROXIMATION

2.1 Motivation

Consider equation (1.1) with $a = 1$ and $b = c = d = e = f = 0$, yielding

$$u_{xx} = g. \tag{2.1}$$

After dropping the unused subscript j, the classical finite differences displayed in equation (1.2) yield the approximation

$$L_h U_i \equiv \frac{1}{h^2}(U_{i+1} - 2U_i + U_{i-1}) = g_i, \tag{2.5}$$

at the point x_i where $U_i = U(x_i)$ and $g_i = g(x_i)$. The operator L_h defined by equation (2.5) is exact if u is in the space $\mathcal{P}_3$ of polynomials with degree at most 3. To see this assume

$$u = a + bx + cx^2 + dx^3.$$

Then at the point $x_i = x + ih$ we get

$$L_h u_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) = 2c + 6dx_i = (u_{xx})_i = g_i$$

where $u_i = u(x_i)$ and $g_i = g(x_i)$.

Equation (2.5), however, only achieves $\mathcal{O}(h^2)$ accuracy. We seek a higher order of accuracy and thus consider an approximation exact on $\mathcal{P}_4$. To do this we introduce an operator I_h and consider the following redefinition of the operator L_h:

$$L_h U_i \equiv \frac{1}{h^2}(\alpha_1 U_{i+1} + \alpha_2 U_i + \alpha_3 U_{i-1}) = \beta_1 g_{i+1} + \beta_2 g_i + \beta_3 g_{i-1} \equiv I_h g_i, \tag{2.6}$$

where α_1, α_2, α_3, β_1, β_2, and β_3 are undetermined weights which define the operators L_h and I_h. Equation (2.5) may be viewed as a special case of equation (2.6) where the approximation is only required to be exact on $\mathcal{P}_3$. To require (2.6) to be exact on $\mathcal{P}_4$ we use the following approach.

Select a basis for $\mathcal{P}_4$ centered at the point x_i:

$$\{p_1, p_2, p_3, p_4, p_5\} = \{1, (x - x_i), (x - x_i)^2, (x - x_i)((x - x_i)^2 - h^2), (x - x_i)^2((x - x_i)^2 - h^2)\}$$

chosen so that at the points x_{i+1}, x_i, and x_{i-1} we have $p_4 = p_5 = 0$. For notational convenience we will henceforth assume that each such stencil is centered at the origin resulting in the simpler notation

$$\{p_1, p_2, p_3, p_4, p_5\} = \{1, x, x^2, x(x^2 - h^2), x^2(x^2 - h^2)\}.$$

Requiring L_h to be exact on $\mathcal{P}_4$ gives

$$L_h p_k = I_h(L p_k)\ ,\ k = 1, \ldots, 5 \tag{2.7}$$

which yields the 5×6 linear system

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & -2 & -2 \\ 0 & 0 & 0 & -6 & 0 & 6 \\ 0 & 0 & 0 & -10 & 2 & -10 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{2.8}$$

This system is consistent and by assuming some convenient normalization such as

$$\beta_1 + \beta_2 + \beta_3 = 1, \tag{2.9}$$

or assuming a value for one of the β's such as

$$\beta_3 = 1, \tag{2.10}$$

the system is reducible and is then equivalent to the linear system

$$\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}.$$

The β's are computed first by solving the system

$$A_3 \beta = r.$$

Then the α's are computed by solving the system

$$A_1\alpha = -A_2\beta.$$

In this case the result using equation (2.10) is

$$\alpha_1 = \alpha_3 = 12, \alpha_2 = -24, \beta_1 = \beta_3 = 1 \text{ , and } \beta_2 = 10 \tag{2.11}$$

which yields the new approximation

$$L_hU_i \equiv \frac{1}{h^2}(U_{i+1} - 2U_i + U_{i-1}) = \frac{1}{12}(g_{i+1} + 10g_i + g_{i-1}). \tag{2.12}$$

Simple Taylor series expansions easily verify that this approximation has $\mathcal{O}(h^4)$ accuracy. Equation (2.12) is known as the Stormer-Numerov formula[2].

Using equation (2.10) to compute (2.12) yields a 5×5 linear system while using equation (2.9) yields a 6×6 linear system. Although equation (2.10) was used in computing the values given in (2.11), it is the normalization of the β's by equation (2.9) that makes the operator I_h a perturbation of the identity operator. This normalization leads to the left hand side of (2.12) being the same as the left hand side of (2.5) as shown in equation (2.12). The left hand sides of standard finite differences and the left hand sides generated by *HODIE* methods are not always the same, and in general, after the normalization of (2.9), *HODIE* methods will generate stencils which differ from classical finite differences by $\mathcal{O}(h)$.

2.2 Generalization

HODIEX is a generalization of the above described procedure. A rectangular $n \times n$ mesh is put over D and at each mesh point an estimate U is obtained for u. To derive the *HODIEX* approximation consider a mesh point (x_i, y_i). There are nine stencil points involved in the equation corresponding to each mesh point, $(x_i + kh, y_i + lh)$, $k, l = -1, 0, 1$. Let U_i, $i = 1, \ldots, 9$, represent the values of the approximation of u at these stencil points. Define (x_j, y_j), $j = 1, \ldots, J$ to be a set of distinct evaluation points (which may or may not include the stencil points) and let $g_j = g(x_j, y_j)$. Then a single *HODIEX* equation is given by

$$L_hU \equiv \frac{1}{h^2}\sum_{i=1}^{9}\alpha_iU_i = \sum_{j=1}^{J}\beta_jg_j \equiv I_hg. \tag{2.13}$$

If only the central stencil point is used as an evaluation point then the method reduces to standard finite differences and achieves $\mathcal{O}(h^2)$ accuracy. It is the use of multiple evaluation points which gives this method high order accuracy. The α's and β's are determined by requiring the approximation to be exact on some finite dimensional vector space in $\mathcal{R}^2$ such as the polynomials $\mathcal{P}_m$ of maximum total degree m. The linear system resulting from equation (2.13) is block tridiagonal, diagonally dominant for sufficiently small h, and banded with half bandwidth $n + 1$.

In general an approximation exact on $\mathcal{P}_m$ will have at most $\mathcal{O}(h^{m-1})$ accuracy, although for certain special cases higher orders are possible. Also, for certain cases, there is a maximum order which may be obtained for the differential operator using a compact nine point stencil and the only solution to equation (2.3) will be the trivial solution. For example the maximum order using such a nine point difference scheme for the Laplacian is $\mathcal{O}(h^6)$. This is because harmonic polynomials exist in $\mathcal{P}_7$ which are zero at all but the central stencil point[7].

The first step in constructing a *HODIEX* approximation is to decide on the desired order; this determines the linear space, the set of basis elements, and the number of evaluation points. For order $\mathcal{O}(h^{m-1})$ we choose $\mathcal{P}_m$ which has $dim = \frac{(m+1)(m+2)}{2}$ basis elements.

The first nine basis elements are the same for all spaces:

$$p_k = \frac{x^r y^s}{h^r h^s} \ , k = 3r + s \ , r, s = 0, 1, 2. \tag{2.14}$$

The remaining basis elements p_{10} through p_{dim} are chosen so that each has a factor $x(x^2 - h^2)$ or $y(y^2 - h^2)$ and thus will equal 0 at the left hand side stencil points. The operators L_h and I_h are computed using these basis functions by forming the linear system

$$L_h p_k = \frac{1}{h^2} \sum_{i=1}^{9} \alpha_i (p_k)_i = \sum_{j=1}^{J} \beta_j (Lp_k)_j = I_h L p_k \ , k = 10, \ldots, dim. \tag{2.15}$$

To solve this system we first select the evaluation points. $J = dim - 9 + 1$ right hand side evaluation points are required since we have 9 stencil points and one normalization equation. Selection of evaluation points will be discussed further in the next section. The above choice of basis functions makes equation (2.15) reducible and we may solve for the α's and β's separately.

The next step is computing the β's by solving the system

$$\sum_{j=1}^{J-1} \beta_j (Lp_k)_j = -(Lp_k)_J, \ k = 10, \ldots dim \tag{2.16}$$

where we have set $\beta_J = 1$. The coefficient matrix in equation (2.15) is always square since it has $(dim - 10) + 1$ rows and $J - 1$ columns and $J - 1 = (dim - 9 + 1) - 1 = (dim - 10) + 1$.

Solving equation (2.16) at every grid point can be very expensive computationally, and special care must be observed in solving for the β's. Although, with the proper choice of evaluation points, the system will always be consistent, it may be singular and thus not have full rank. In *HODIE* a very fast solve is obtained for $\mathcal{P}_4$; if $order = 6$ is requested and the PDE meets the previously mentioned criteria, then an update to the $\mathcal{P}_4$ solution gives the $\mathcal{P}_5$ solution. Since *HODIEX* can use any of the polynomial spaces $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, ... with a general second order linear two dimensional PDE, a more general solve is used.

In an early version of *HODIEX* this system was solved by a least squares solution generated through a QR factorization, however the consistency of equation (2.15) allows the present version of *HODIEX* to obtain a more precise solution by using Gaussian elimination with full or partial pivoting. Since achieving a higher order requires going to a larger linear space which results in a larger linear system to solve for the βs, the tradeoff for higher order is the resulting increase in computation time.

After evaluating the β's, the α's are computed by solving the system

$$\frac{1}{h^2} \sum_{i=1}^{9} \alpha_i (p_k)_i = \sum_{j=1}^{J} \beta_j (Lp_k)_j \ , k = 1, \ldots, 9 \ . \tag{2.17}$$

In solving for the α's notice that the coefficient matrix in the linear system is

$$\begin{pmatrix} (p_1)_1 & (p_1)_2 & \cdots & (p_1)_9 \\ (p_2)_1 & (p_2)_2 & \cdots & (p_2)_9 \\ \vdots & \vdots & \ddots & \vdots \\ (p_9)_1 & (p_9)_2 & \cdots & (p_9)_9 \end{pmatrix}. \tag{2.18}$$

In fact the basis functions were so chosen that this matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{2.19}$$

and the solution can be computed directly (without factoring the matrix) given a right hand side. Note also that this system is independent of the operator L and the central stencil point.

At this point we have determined the operators L_h and I_h. Unless we have constant coefficients in the PDE, this procedure must be carried out at each grid point. The final step is to solve the system

$$L_h U_{i,j} = I_h g_{i,j},\ i,j = 1,\ldots,n \tag{2.19}$$

to obtain the estimates U of u. Equation (2.19) is just the linear system described by equation (1.3).

2.3 Selection of Evaluation Points

The selection of evaluation points is a critical factor in obtaining the maximum possible order for any particular linear space. An improper choice of evaluation points can reduce the order by levels of magnitude. The evaluation point stencils used by *HODIEX* for $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, and $\mathcal{P}_7$ are shown below:

$$\mathcal{P}_4 : \begin{pmatrix} & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ & \bullet & \bullet \end{pmatrix}, \quad \mathcal{P}_5 : \begin{pmatrix} \bullet & & \bullet & & \bullet \\ & & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ & & \bullet & & \\ \bullet & & \bullet & & \bullet \end{pmatrix},$$

$$\mathcal{P}_6 : \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & & \bullet & & \bullet \\ \bullet & \bullet & & \bullet & \bullet \\ \bullet & & \bullet & & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}, \quad \mathcal{P}_7 : \begin{pmatrix} \bullet & \bullet & & \bullet & & \bullet & \bullet \\ \bullet & \bullet & & \bullet & & \bullet & \bullet \\ & & & \bullet & & & \\ \bullet & \bullet & \bullet & & \bullet & \bullet & \bullet \\ & & & \bullet & & & \\ \bullet & \bullet & & \bullet & & \bullet & \bullet \\ \bullet & \bullet & & \bullet & & \bullet & \bullet \end{pmatrix}.$$

In order to maximize symmetry, the central stencil point is not used for $\mathcal{P}_6$ and $\mathcal{P}_7$. In all cases the stencils shown lie within the standard 9-point compact finite difference stencil. An obvious cost of *HODIE* methods is the extra evaluations of the source term g and the PDE coefficients. Since these stencils overlap as the *HODIEX* discretization moves from grid point to grid point, many of these evaluations can be re-used. The current version of *HODIEX* does 3 evaluations of g per grid point for $\mathcal{P}_4$, 6 for $\mathcal{P}_5$, 9 for $\mathcal{P}_6$, and 14 for $\mathcal{P}_7$. The tradeoff for a more efficient implementation using less evaluations is more memory to store results.

3. TEST PROBLEMS

Four test problems were selected. The first two problems have no cross derivatives terms while one problem has variable coefficients and the other has constant coefficients. Both of the second two problems have cross derivatives terms, and again one problem has variable coefficients while the other has constant coefficients.

Each test problem has the same domain, the unit square, with Dirichlet boundary conditions. We chose the function $g(x,y)$ so that the true solution is known. The four problems chosen are as follows:

Problem 1.

This problem is general with variable coefficients but no cross derivative term. The PDE operator is self adjoint and

$$Lu = (e^{xy}u_x)_x + \left(e^{-xy}u_y\right)_y - \frac{u}{1+x+y}$$

which has the true solution

$$u(x,y) = 0.75e^{xy}\sin(\pi x)\sin(\pi y).$$

Problem 2.

This problem is Poisson's equation

$$Lu = u_{xx} + u_{yy}$$

with true solution

$$u(x,y) = 3e^{x+y}\left(x - x^2\right)\left(y - y^2\right).$$

Problem 3.

This problem has a cross derivative term and variable coefficients with

$$Lu = au_{xx} + bu_{xy} + cu_{yy},$$

where $a = 1 + u_x^2$, $b = -2u^2$, and $c = 1 + u_y^2$, and with true solution

$$u(x,y) = e^{x+y}.$$

Problem 4.

Like Problem 3, this problem has a cross derivative term, but the coefficients are constant. We have

$$Lu = 4u_{xx} - u_{xy} + 4u_{yy},$$

with true solution is

$$u(x,y) = (x - 3y)^2 e^{x-y}.$$

4. PERFORMANCE RESULTS

Each of the test problems are run for n = 8, 16, 32, 32, and 128. In each case *HODIEX* is computed for $\mathcal{P}_4$ through $\mathcal{P}_7$ even though the order may no longer be increasing. For each problem *9 point star* is run; *HODIE* is only run on Problem 1 and Problem 2 which have no cross derivative term. Problem 2 also allows *HODIE* to be run with *order* = 6 for $\mathcal{O}(h^6)$ accuracy.

In the tables "disc" is the discretization time while "total" is the total time including band Gauss elimination. We use the infinity norm to compute the error (the maximum difference between the true solution and the approximate solution at each grid point). *Order* is computed by comparing the reduction in the infinity norm of the error in one step with the error from the previous step. The step size is being reduced by $\frac{n_1+1}{n_2+1}$ where n_1 and n_2 are successive numbers of interior grid points, hence we have

$$\left(\frac{n_1+1}{n_2+1}\right)^r = \left(\frac{e_2}{e_1}\right) \tag{4.1}$$

where e_1 and e_2 are the errors on two succeeding steps and where r is the order. Hence

$$r = \frac{(\log e_2 - \log e_1)}{\log(n_1+1) - \log(n_2+1)}. \tag{4.2}$$

9 point star achieves $\mathcal{O}(h^2)$ accuracy on Problem 1 while both *HODIEX* and *HODIE* achieve $\mathcal{O}(h^4)$. However, notice that *HODIEX* achieves a slightly better accuracy using $\mathcal{P}_5$, and then the accuracy drops off for $\mathcal{P}_6$, or $\mathcal{P}_7$ indicating maximum order is achieved on $\mathcal{P}_5$.

9 point star again achieves $\mathcal{O}(h^2)$ accuracy on Problem 2. *HODIE* achieves $\mathcal{O}(h^4)$ and on this problem *HODIE* may be run with an option for *order* = 6 which yields $\mathcal{O}(h^6)$ accuracy. *HODIEX* is very consistent achieving $\mathcal{O}(h^3)$, $\mathcal{O}(h^4)$, $\mathcal{O}(h^5)$, and $\mathcal{O}(h^6)$ respectively on $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, and $\mathcal{P}_7$. Also note that *HODIEX*'s $\mathcal{P}_6$ and $\mathcal{P}_7$ are faster than *HODIE*'s sixth order while obtaining the same accuracy.

Again *9 point star* achieves $\mathcal{O}(h^2)$ accuracy on Problem 3. *HODIE* cannot be run on this problem due to the cross derivative term. *HODIEX* performs well on $\mathcal{P}_4$ and $\mathcal{P}_5$ achieving $\mathcal{O}(h^3)$ and $\mathcal{O}(h^4)$ respectively, but with little improvement for $\mathcal{P}_6$ or $\mathcal{P}_7$.

Finally, for Problem 4 that *9 point star* again achieves $\mathcal{O}(h^2)$ accuracy. *HODIE* cannot be run on this problem due to the cross derivatives. *HODIEX* performs well on $\mathcal{P}_4$ and $\mathcal{P}_5$ achieving $\mathcal{O}(h^3)$ and $\mathcal{O}(h^4)$ respectively, but beyond $\mathcal{P}_5$ there is no improvement (data not shown).

These results are shown on the attached plots. For each problem there are two plots, one showing achieved order and the other a log-log plot of total time versus error.

5. CONCLUSIONS

On the general problem with no cross derivatives, *HODIEX* and *HODIE* achieved nearly identical accuracy while *HODIE* runs faster. However, on problems with constant coefficients and no cross derivatives *HODIEX* runs faster while *HODIEX* and *HODIE* achieved similar accuracy. Both are greatly superior to *9 point star* requiring much less time to achieve the same accuracy.

On general problems with cross derivatives, *HODIEX* consistently achieved at least $\mathcal{O}(h^4)$ and was much faster than *9 point star* with the same number of grid points on problems with constant coefficients. *HODIEX* also ran faster than *9 point star* on the problem with variable coefficients to obtain similar accuracy. *HODIE* of course will not run on these problems.

TABLE 1: Data Problem 1. $g(x,y) = (e^{xy}u_x)_x + (e^{-xy}u_y)_y - \frac{u}{1+x+y}$

	HODIEX $\mathcal{P}_4$			HODIEX $\mathcal{P}_5$			HODIEX $\mathcal{P}_6$		
n	error	disc	total	error	disc	total	error	disc	total
8	$6.1E-5$	.32	.37	$1.6E-5$	1.15	1.20	$6.9E-4$	3.00	3.05
16	$4.9E-6$	1.58	2.20	$1.2E-6$	4.60	5.22	$6.7E-5$	12.20	12.83
32	$3.4E-7$	5.37	12.89	$8.5E-8$	19.13	26.65	$4.7E-5$	48.86	56.36
64	$2.2E-8$	23.12	129.97	$5.6E-9$	73.28	180.45	$4.8E-7$	200.68	307.28
128	$1.4E-9$	98.76	1873.13	$3.6E-10$	298.67	2075.80	$6.5E-8$	798.05	2574.55
	HODIEX $\mathcal{P}_7$			9 Point Star			HODIE		
n	error	disc	total	error	disc	total	error	disc	total
8	$4.9E-3$	7.15	7.20	$8.6E-3$	.53	.56	$6.4E-5$	.87	.94
16	$4.0E-5$	28.70	29.33	$2.5E-3$	.93	1.03	$5.1E-6$	1.47	2.10
32	$1.9E-6$	114.61	122.11	$6.5E-4$	1.73	9.03	$3.6E-7$	3.77	11.27
64	$2.6E-6$	460.46	567.31	$1.7E-4$	4.75	109.33	$2.4E-8$	12.88	119.03
128	$3.0E-6$	1824.64	3601.84	$4.3E-5$	16.72	1757.87	$1.6E-9$	47.70	1811.45

For good results, the selection of evaluation points is critical. We tried various selections on the linear spaces above $\mathcal{P}_4$, but no conclusions were reached pertaining to the optimal points. The optimal selection of the evaluation points is problem dependent. *HODIEX* allows evaluation points to be easily changed by just choosing coordinates.

Although no results for spaces higher than $\mathcal{P}_7$ are given in this paper, *HODIEX* can be run on these spaces. However, obtaining order above $\mathcal{O}(h^6)$ appears to have questionable value. On those problems for which the discretization exists, the error already approaches machine epsilon with 100 grid points.

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TABLE 2: Data Problem 2. $g(x, y) = u_{xx} + u_{yy}$

	HODIEX $\mathcal{P}_4$			HODIEX $\mathcal{P}_5$			HODIEX $\mathcal{P}_6$		
n	error	disc	total	error	disc	total	error	disc	total
8	$1.4E-5$	.10	.17	$1.0E-4$	.12	.19	$8.6E-6$	.20	.27
16	$1.1E-6$	.17	.79	$6.4E-8$	.40	1.03	$3.4E-7$	.70	1.42
32	$7.9E-8$	.78	8.28	$3.4E-9$	1.10	8.60	$1.3E-8$	2.27	9.99
64	$5.2E-9$	3.17	110.14	$2.9E-10$	4.43	111.58	$5.5E-12$	7.98	115.13
128	$3.3E-10$	12.80	1790.15	$5.1E-11$	18.17	1796.62	$2.8E-14$	32.57	1808.19
	HODIEX $\mathcal{P}_7$			9 Point Star			HODIE		
n	error	disc	total	error	disc	total	error	disc	total
8	$1.2E-7$	.30	.37	$3.5E-3$	.48	.53	$4.3E-5$	.73	.78
16	$2.6E-9$	.90	1.53	$9.7E-4$	.80	.90	$3.4E-6$	.95	1.57
32	$1.0E-10$	2.83	10.33	$2.6E-4$	1.18	8.48	$2.4E-7$	1.78	9.28
64	$2.9E-12$	9.54	115.70	$6.7E-5$	2.58	109.43	$1.6E-8$	5.07	113.55
128	$1.4E-14$	38.72	1821.00	$1.7E-5$	8.15	1794.22	$1.1E-9$	18.10	1817.43
	HODIEX (order=6)								
n	error	disc	total						
8	$6.6E-8$	.80	.85						
16	$1.5E-9$	1.22	1.84						
32	$2.8E-11$	2.90	10.40						
64	$4.7E-13$	9.50	117.98						
128	$2.5E-14$	40.97	1840.80						

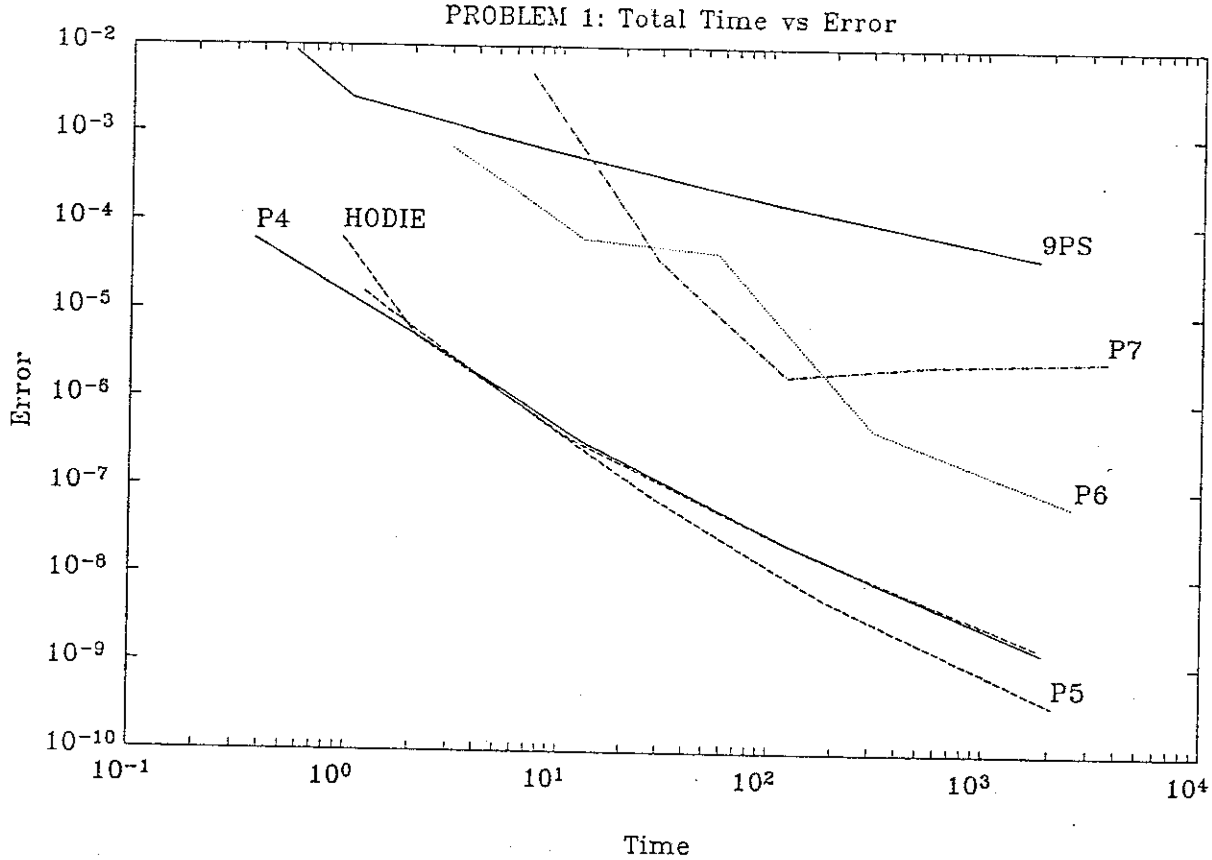
TABLE 3: Data Problem 3. $g(x, y) = au_{xx} + bu_{xy} + cu_{yy}$

	HODIEX $\mathcal{P}_4$			HODIEX $\mathcal{P}_5$			HODIEX $\mathcal{P}_6$		
n	error	disc	total	error	disc	total	error	disc	total
8	$8.5E-6$	.27	.32	$9.6E-8$	1.00	1.05	$9.4E-9$	3.07	3.12
16	$1.5E-6$	1.38	2.03	$7.6E-9$	4.65	5.27	$8.6E-10$	12.55	13.17
32	$2.2E-7$	5.83	13.33	$5.3E-10$	18.47	25.95	$7.4E-11$	50.32	57.80
64	$3.0E-8$	23.90	130.78	$3.5E-11$	74.34	181.16	$5.5E-12$	198.83	305.54
128	$3.9E-9$	90.54	1886.49	$2.9E-12$	294.27	2070.95	$1.0E-12$	797.32	2580.47
	HODIEX $\mathcal{P}_7$			9 Point Star					
n	error	disc	total	error	disc	total			
8	$2.0E-8$	7.32	7.37	$1.4E-3$	.73	.80			
16	$3.7E-9$	28.93	29.25	$3.9E-4$	.93	1.55			
32	$4.0E-11$	116.11	123.72	$1.1E-4$	1.67	9.19			
64	$1.4E-12$	449.31	556.62	$2.8E-5$	4.57	112.15			
128	$9.8E-13$	1805.50	3588.13	$7.0E-6$	16.00	1803.35			

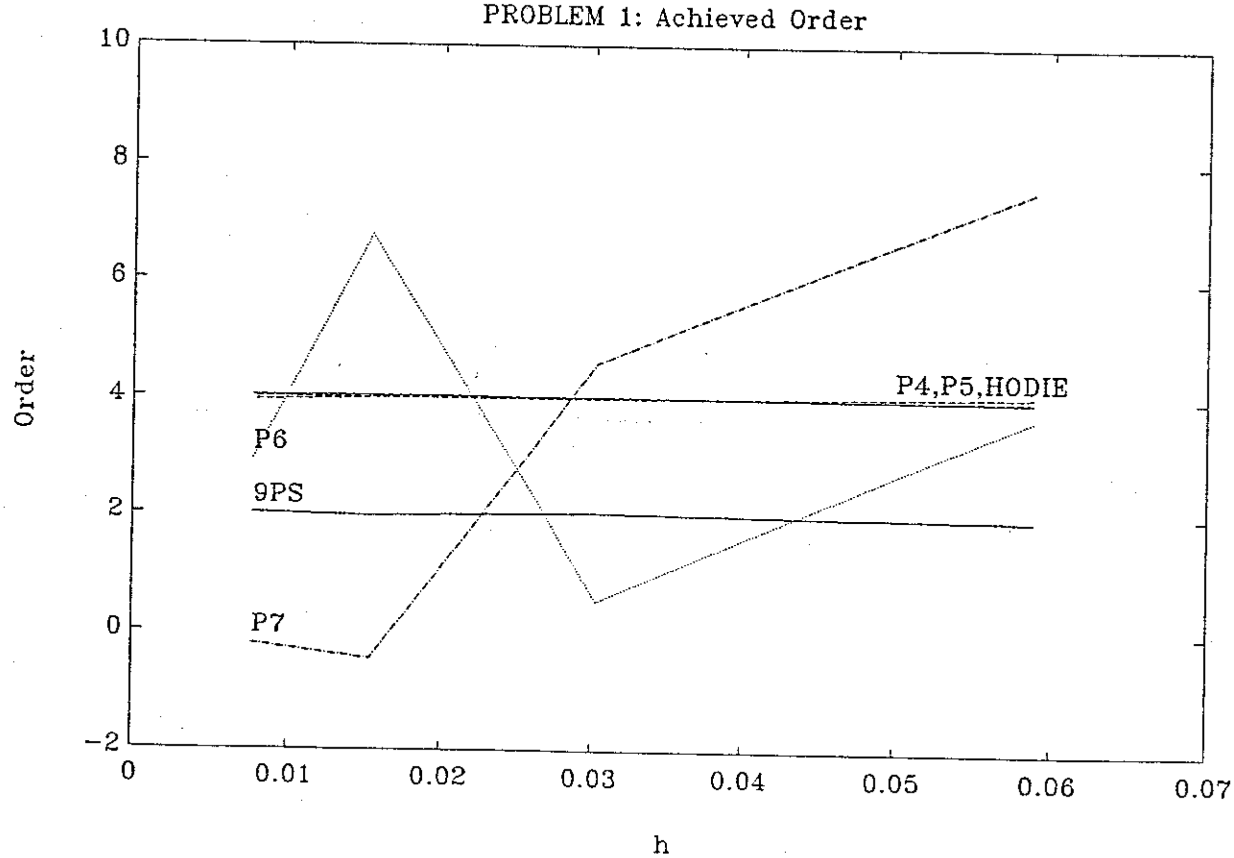
TABLE 4: Data Problem 4. $g(x,y) = 4u_{xx} - u_{xy} + 4u_{yy}$

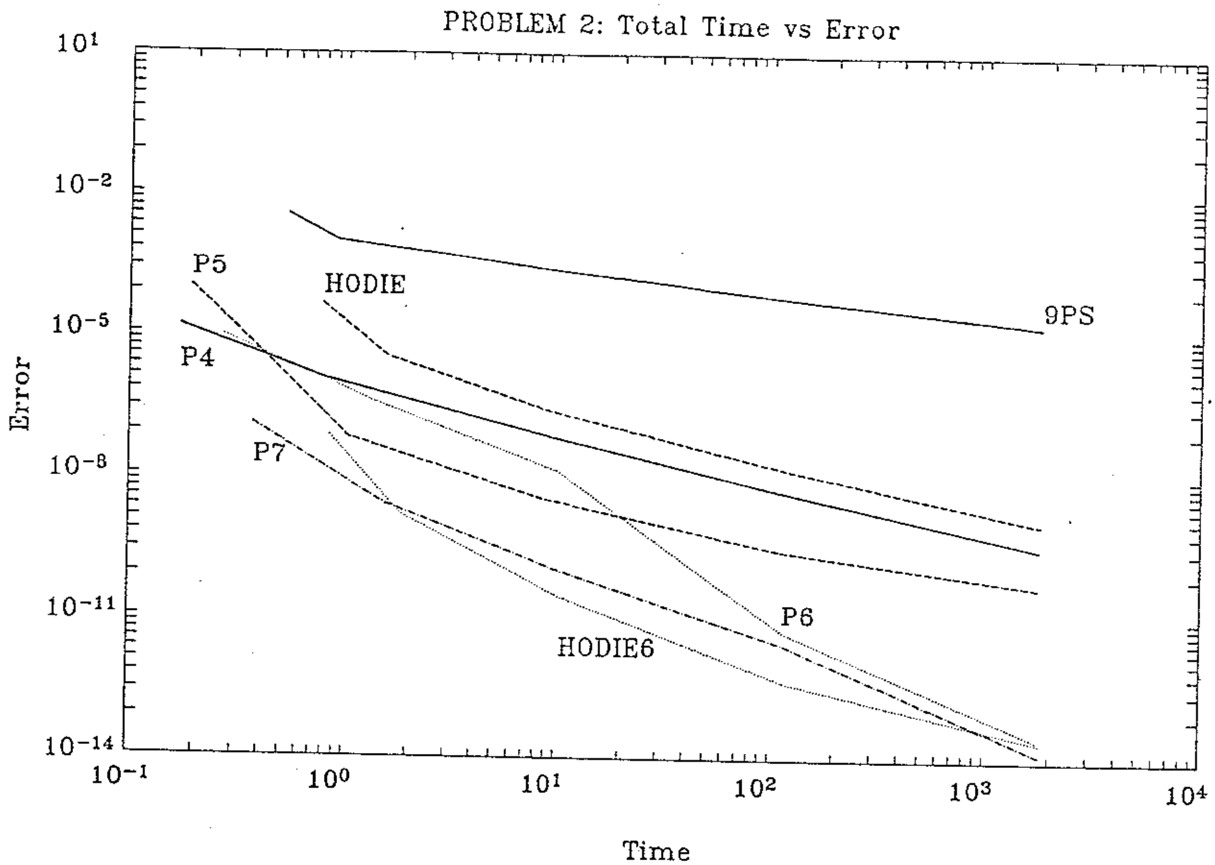
	HODIEX $\mathcal{P}_4$			HODIEX $\mathcal{P}_5$			9 Point Star		
n	error	disc	total	error	disc	total	error	disc	total
8	$1.0E-5$	.07	.12	$5.2E-6$	.12	.17	$1.1E-2$	.63	.68
16	$1.3E-6$	.17	.80	$4.1E-7$	.37	.99	$3.2E-3$	1.17	1.79
32	$1.5E-7$	.82	8.32	$2.9E-8$	1.38	8.76	$8.5E-4$	2.58	10.08
64	$1.9E-8$	3.27	110.42	$1.9E-9$	4.88	111.88	$2.2E-4$	8.18	115.56
128	$2.3E-9$	13.53	1790.21	$1.2E-10$	20.60	1797.80	$5.5E-5$	32.90	1818.62

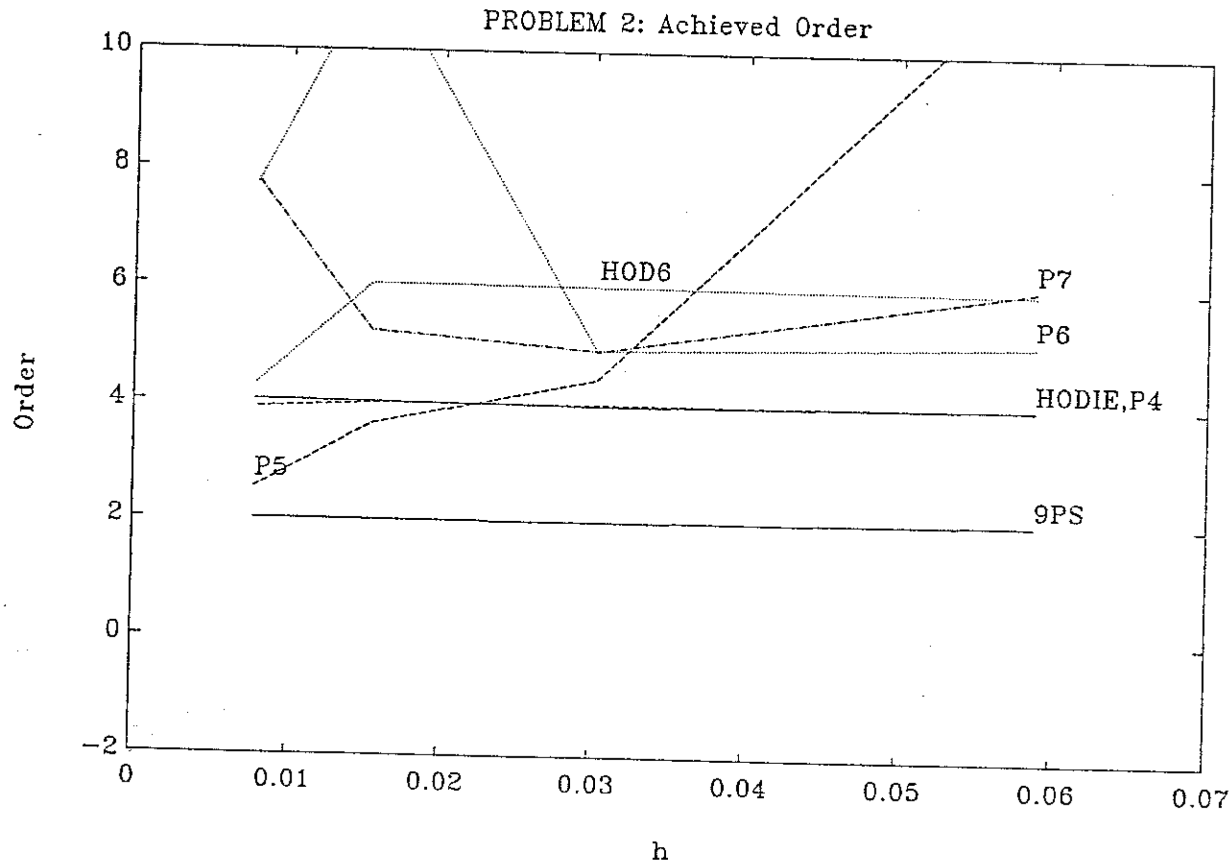
PROBLEM 1: Total Time vs Error



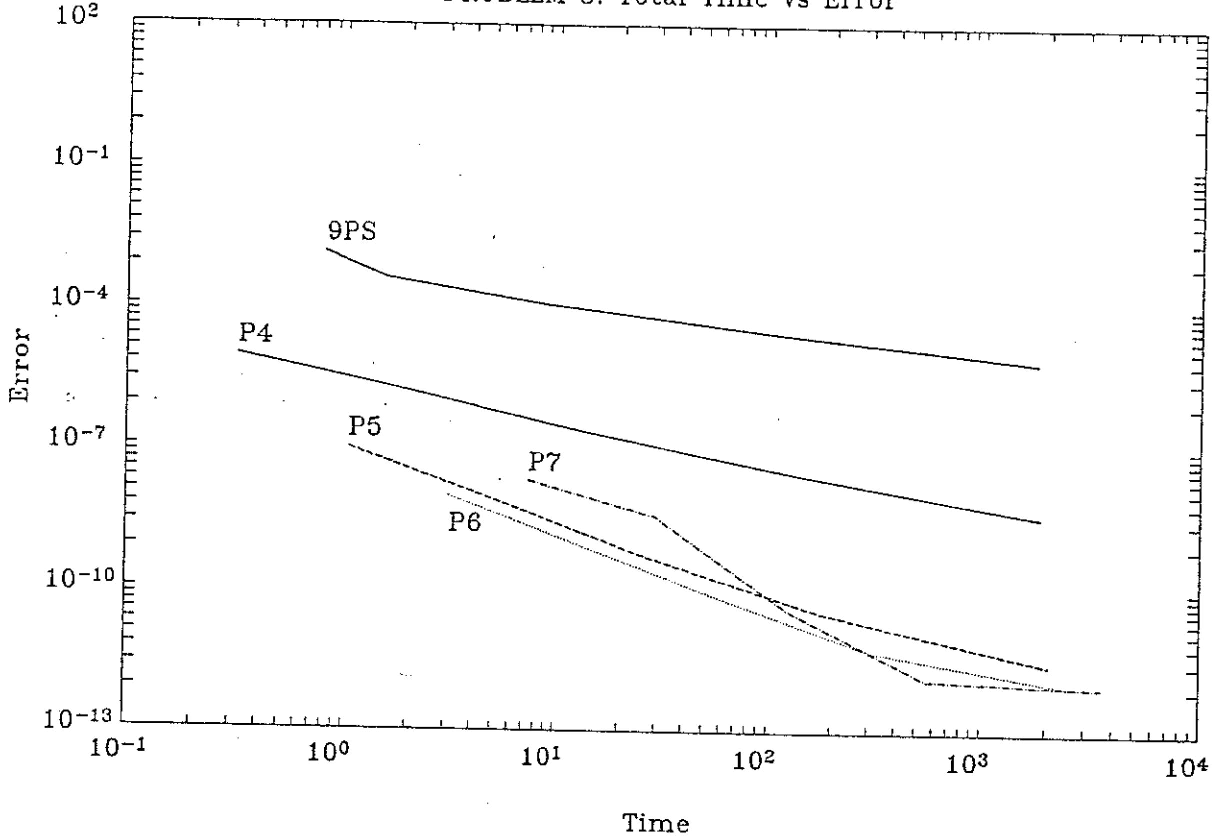
PROBLEM 1: Achieved Order



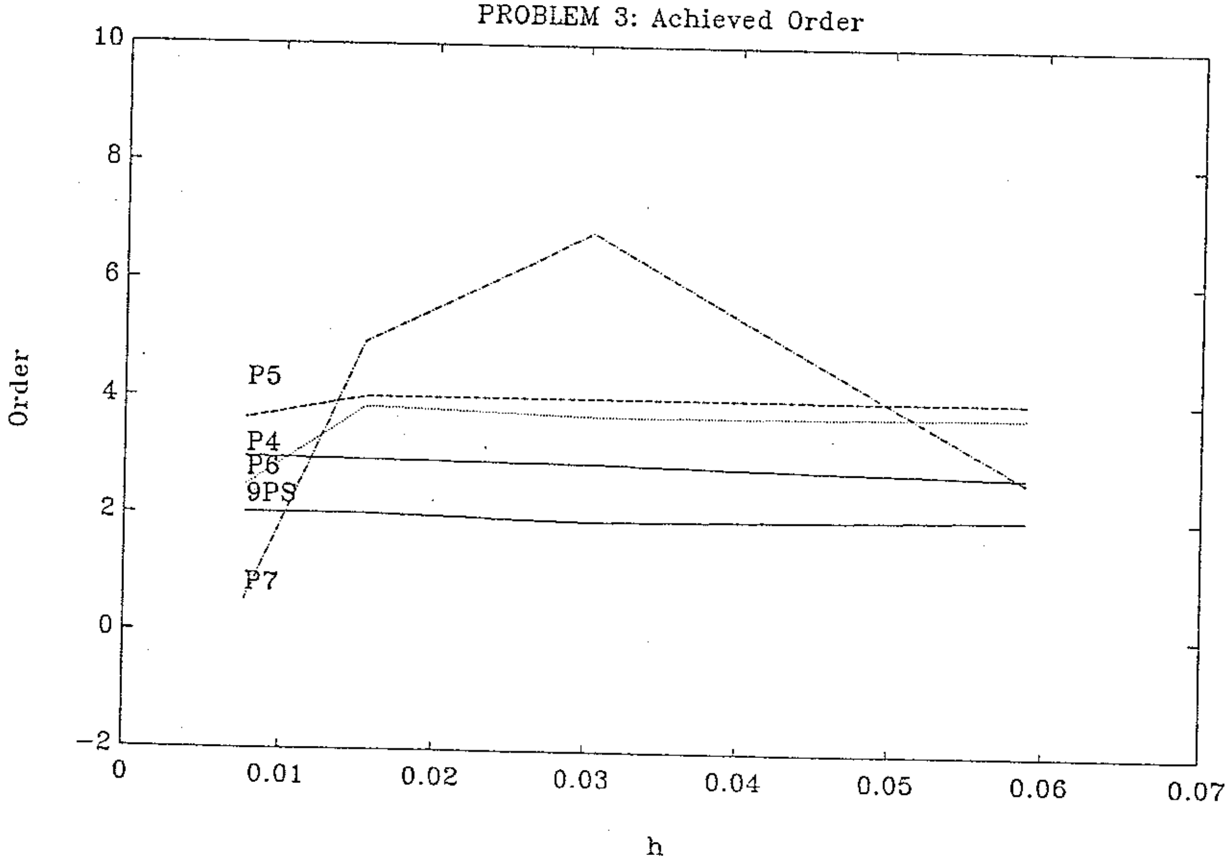

PROBLEM 2: Total Time vs Error
Error
Time
10^{1}
10^{-2}
10^{-5}
10^{-8}
10^{-11}
10^{-14}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
P5
HODIE
9PS
P4
P7
P6
HODIE6

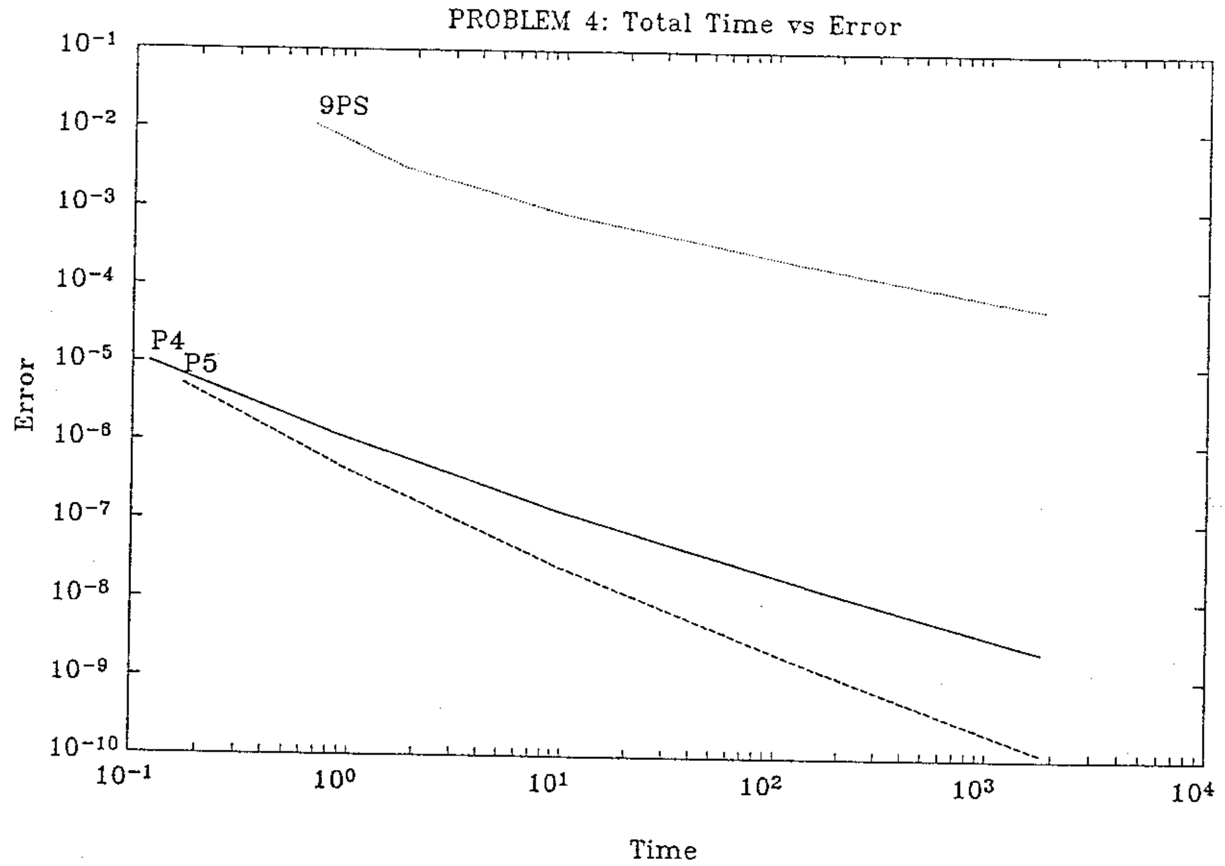

PROBLEM 2: Achieved Order
Order
h
10
8
6
4
2
0
−2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
HOD6
P7
P6
HODIE,P4
P5
9PS

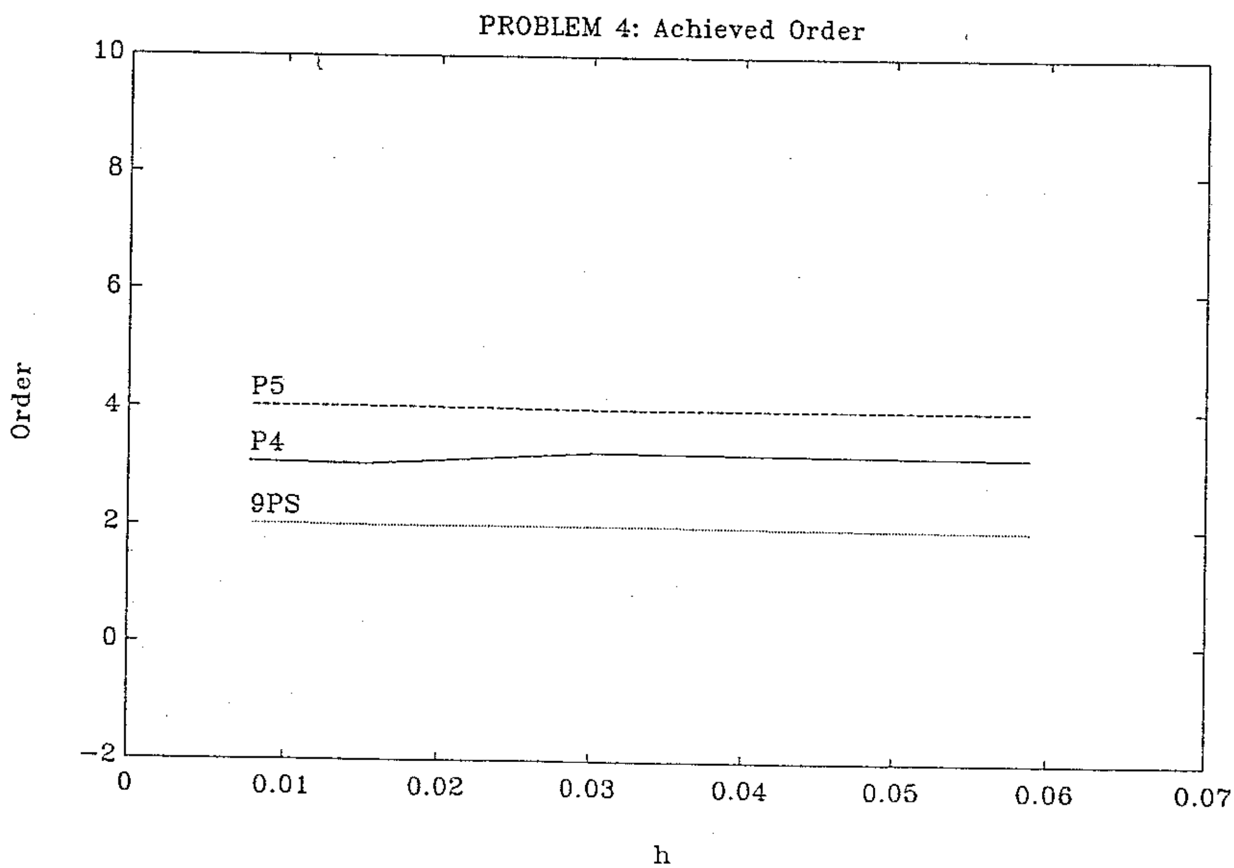
PROBLEM 3: Total Time vs Error



PROBLEM 3: Achieved Order




PROBLEM 4: Total Time vs Error
9PS
P4
P5
Error
Time
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}


PROBLEM 4: Achieved Order
P5
P4
9PS
Order
h
10
8
6
4
2
0
−2
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07