Representing Polyhedra: Faces are Better than Vertices

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Abstract

In this paper, we investigate the reconstruction of planar-faced polyhedra given their spherical dual representation. We prove that the spherical dual representation is unambiguous for all genus 0 polyhedra and that a genus 0 polyhedron can be uniquely reconstructed in polynomial time. We also prove that when the degree of the spherical dual representation is at most four, the representation is unambiguous for polyhedra of any genus. The first result extends the well known result that a vertex or face connectivity graph represents a polyhedron unambiguously when the graph is triconnected and planar in the case of planar-faced polyhedra. The second result shows that when each face of a polyhedron of arbitrary genus has at most four edges, the polyhedron can be reconstructed uniquely. This extends the result that a polyhedron can be uniquely reconstructed when each face of the polyhedron is triangular. To obtain this result, we prove that the 4-dimensional hypercube, a classic example of ambiguity in the wire frame representation scheme, is unambiguous when the same connectivity graph is viewed as the spherical dual representation of a polyhedron and thus that faces are a better representation than vertices. A result of the reconstruction algorithm is that high level features of the polyhedron are naturally extracted. Both of our results explicitly use the fact that the faces of the polyhedron are planar. We conjecture that the spherical dual representation is unambiguous for polyhedra of any genus.

Keywords: polyhedra, reconstruction, computer vision, CAD

1 Introduction

A common means of representing a 3-dimensional object is through the abstraction known as a polyhedron. A *polyhedron* is a closed surface that partitions Euclidean 3-space E^3 into 3 sets: (i) points lying inside the polyhedron, (ii) points of the polyhedron, and (iii) points outside the polyhedron. The boundary of a polyhedron is a two-manifold. Viewed combinatorially, the surface consists of faces, edges, and vertices. Each edge is shared by exactly two faces, and each edge has exactly two vertices as its endpoints. The edges incident to any vertex appear on the surface in a cyclic order around the vertex. Alternately, the faces incident to the vertex appear in a cyclic order around the vertex, and two faces adjacent in the order share an edge incident to the vertex. We only consider polyhedra with planar faces, that is, each face is contained in a plane.

Constructive solid geometry views a polyhedron in terms of point sets while boundary representations, as the name indicates, characterize a polyhedron based on its boundary [17]. Since the boundary consists of entities of various dimensionals—faces, edges, and vertices—there are various schemes for representing a polyhedron. Obtaining a polyhedron from its representation is termed *reconstruction.* If there is always a single polyhedron that can be obtained from any representation in the representation scheme, the polyhedron is *uniquely reconstructible* in that representation scheme. The problem we consider in this paper is the unique reconstruction of polyhedra in a representation scheme that encodes the minimum amount of information required for reconstruction and that is useful for computer vision and other applications.

The spherical dual representation [18] is a representation scheme for polyhedra useful in both solid modeling and computer vision. The spherical dual representation of a polyhedron is a graph in which each face of the polyhedron is a node and is labeled by the equation of the plane containing the face. A node is connected by an arc to another if the two faces share an edge in the polyhedron. No ordering of the arcs around each node is specified. In fact, no explicit order information whatsoever is maintained. The spherical dual representation scheme can be viewed as the dual of the wire frame representation of polyhedra. Roach, Wright, and Ramesh [18] raise, but do not answer, the question of unique reconstructibility for this representation scheme.

In this paper, we investigate the reconstruction of a polyhedron from its spherical dual representation. It is well known that the wire frame representation (i.e., the vertex connectivity graph) of a polyhedron is ambiguous [14,17]. Also, given either the vertex or face connectivity graph of a genus 0 polyhedron, algorithms are known to *uniquely* reconstruct it only when the graph is triconnected [6]. We extend these algorithms to uniquely reconstruct any genus 0 polyhedron, given its face connectivity graph (spherical dual representation). We also show that the face connectivity graph is not exactly the dual of the vertex connectivity graph with regard to ambiguity. This is accomplished by an example (the four-dimensional hypercube) which is ambiguous as a vertex connectivity graph, but which is unambiguous as a spherical dual representation. We then generalize this argument to prove that all spherical dual representations of at most degree 4 represent polyhedra unambiguously. The results have an added importance since the spherical dual representation also has some interesting applications in computer vision. For example, the *SDR* provides some useful relationships between the representation of an object and its image under perspective projection [16].

The structure of the paper is as follows. The next section contains the necessary graph theoretic and topological definitions. Section 3 reviews previous work in solid modeling representation and reconstruction. In Section 4, we develop our algorithm for uniquely reconstructing a genus 0 polyhedron given its SDR. Section 5 proves that the hypercube is *not* an ambiguous representation as an SDR. Section 6 extends that result to the SDR of any polyhedron having maximum degree 4. The last section concludes with observations and conjectures.

2 Terms and Definitions

A *graph* $G = (N, A)$ consists of a set N of *nodes* and a set A of *arcs;* each arc is an unordered pairs of distinct elements from N. (We have chosen this non-standard terminology for undirected graphs—nodes and arcs instead of vertices and edges—to avoid confusion between the vertices and edges of a polyhedron and the nodes and arcs of the associated SDR graph. Nodes and arcs are generally used in the context of directed graphs.) If A is a multiset, that is, if any arc may occur several times, then G is a *multigraph*. Multiple arcs between the same pair of nodes are called *parallel arcs*.

A *path* P between nodes v_0 and v_k in a graph G is a sequence of nodes $v_0, \ldots, v_k$ such that $(v_{i-1}, v_i) \in A$, $i = 1, \ldots, k$. Path P is a *simple path* if $v_0, \ldots, v_k$ are distinct. A *cycle* C in G is a path $v_0, \ldots, v_k$ such that $v_0 = v_k$. Cycle C is a *simple cycle* if $v_0, \ldots, v_{k-1}$ are distinct. A graph $G = (N, A)$ is *connected* if there exists a path between every pair of nodes in N. The number of arcs incident on a node v_i is called the *degree* of the node. Two arcs are said to be in *series* if they have exactly one node in common and if this node is of degree two. A node $v \in N$ is an *articulation point* of a connected graph $G = (N, A)$ if the subgraph induced by $N - \{v\}$ is not connected. A connected graph G is *biconnected* if G contains no articulation point. A *biconnected component* of G is a maximal induced subgraph of G which is biconnected. Let v_1, v_2 be a pair of nodes of a biconnected graph $G = (N, A)$; $\{v_1, v_2\}$ is a *separation pair* for G if the induced subgraph on $N - \{v_1, v_2\}$ is not connected. A biconnected graph G is *triconnected* if G contains no separation pair. A *triconnected component* of G is a maximal induced subgraph of G which is triconnected. Hopcroft and Tarjan [8] give an algorithm to find the triconnected components of a graph in time linear in the size of the graph.

The *genus* of an orientable, compact surface is the maximum number of non-intersecting simple closed curves that can be removed from its surface without disconnecting it. Thus the genus of a sphere is 0, and the genus of a torus is 1. In general, an orientable surface with g holes has genus g. The *genus* of a polyhedron is the genus of its surface.

A graph G is said to be topologically *embedded* in a surface S when it is drawn on S such that no two arcs intersect except at their common nodes (see Gross and Tucker [4]). If a graph is embedded in a surface, the complement of its image is a finite set of regions. A *face* of a topological embedding of G is a connected component of the complement of the image of G. (Henceforth, we use *facets*, rather than faces, to refer to the two-dimensional components of a polyhedron. We reserve *faces* to refer to the regions of a graph embedding.) The *genus* of a graph G is the genus of the orientable surface S of least genus such that G can be topologically embedded in S. A graph G is *planar* if G has an embedding in a plane (or, equivalently, in a sphere).

Given a graph G, a *rotation* of a node v of G is a cyclic permutation of all arcs incident with v. A *rotation* of G consists of a rotation for every node of G. Every rotation uniquely determines an embedding of G in some orientable surface [4].

The *boundary* of a face f is the set of arcs in the closure of f. Two embeddings of a graph are *equivalent* when the boundary of a face in one embedding always corresponds to the boundary of a face in the other. The embedding of a graph on a surface is said to be *unique* if all its embeddings in that surface are equivalent. A planar graph has a unique embedding if and only if it is triconnected [23].

3 Representations

In this section, we review some representations that have been used in geometric modeling and in computer vision, including the spherical dual representation. We also review known techniques for reconstructing a polyhedron from its representations and some preliminaries on embedding graphs in surfaces.

3.1 Representations in Geometric Modeling

Geometric modeling is the art of creating data structures and algorithms capable of representing and calculating the three-dimensional physical shape of an object [14]. Some important characteristics of a representation scheme for geometric modeling [17] that have theoretical and practical implications are:

1. *Domain:* The domain of a representation scheme characterizes the descriptive power of the scheme; the domain is the set of entities representable in the scheme.

2. *Validity:* The range of a representation scheme is the set of representations which are *valid*, that is, represent an actual "solid." A representation scheme is said to be *valid* if every representation in the scheme is valid.

3. *Completeness:* A representation is *unambiguous* if it corresponds to a single object. A representation scheme is *complete* if all of its valid representations are unambiguous.

4. *Uniqueness:* A representation of a solid is *unique* if it is the *only* representation for the solid in the scheme. A representation scheme is *unique* if all its valid representations are unique.

In this paper, we concentrate on the issue of whether a representation scheme is complete. As we are attempting to give a representation scheme using the least possible amount of information that yields an unambiguous representation for each object, completeness is the central characteristic considered here and the most difficult to prove.

The following are some of the common solid modeling schemes [14,17,20].

3.1.1 *Wire frame representation*

A wire frame model represents a solid object by representing its vertices and edges only. Each edge is typically represented by a six-tuple

$$< x_1, y_1, z_1, x_2, y_2, z_2 >$$

giving the coordinates of the two endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2) of the edge.

The main drawback of this representation scheme is its ambiguity. A wire frame model in general does not have enough information to represent an object uniquely (see Section 5). Characterizing it another way, two or more different objects can have the same set of edges. Thus wire frame representation is not a *complete* representation scheme.

3.1.2 *Constructive Solid Geometry*

The most general form of the constructive solid geometry (CSG) approach is the *half-space* model. In this model, solids are represented by a finite number of simple point sets called half-spaces that are combined by the standard set operations of *union, intersection* and *difference.*

The CSG representation scheme is complete but not unique.

3.1.3 *Boundary Representations*

Boundary representations represent a solid object by storing a description of its boundary. The boundary of an object divides space into two parts, one with finite volume and the other having infinite volume. If we assume that all objects have finite volume, then an object can be represented unambiguously by its boundary. The boundary is divided into a three-level hierarchy of entities: *facets, edges,* and *vertices.*

A widely used boundary representation is the solid modeling scheme based on Euler operators. Euler's formula for a convex polyhedron gives a relationship among the number f of facets, the number e of edges, and the number v of vertices of a convex polyhedron: $v - e + f = 2$. Euler's formula is generalized to an arbitrary polyhedron by introducing three additional parameters

1. The total number r of internal loops in the facets of the solid,

2. The genus g of the solid, and

3. The number s of disconnected components in a solid with a disconnected surface.

The general Euler formula is $v - e + f = 2 * (s - g) + r$.

The operations used to construct the representation are called Euler operators because every operator used satisfies the Euler equation (e.g., two Euler operators are *mev*, for make edge vertex, and *kef,* for kill edge, face). It has been proved [13] that Euler operators are sound and complete, i.e., they create only meaningful models and every meaningful model can be constructed by them. Similar to the *CSG* scheme, boundary representations based on Euler operators are complete but not unique.

Representation schemes which are *both* unambiguous and unique are highly desirable because they are one-to-one mappings from the object space to the representation space. This implies that distinct representations in such schemes correspond to distinct objects, and therefore object equality may be determined by algorithms which compare object representations "syntactically" [17]. Both the *CSG* scheme and the boundary representation scheme are nonunique.

3.2 Representations in Computer Vision

Object representations in computer vision are generally surface based. We review some of the representation schemes used in computer vision and introduce the spherical dual representation.

The Gaussian sphere provides a method for representing the surface of an object by the orientations of its facets. Imagine the set of unit normal vectors associated with the facets of a polyhedron, all translated to a common origin. The locus of the end points of these vectors (all on the unit sphere) forms the Gaussian image of the polyhedron. The Gaussian image represents the orientation of the object only. Size and shape information is lost, making it impossible to reconstruct the object from the Gaussian image. A popular extension of the Gaussian image representation is the *extended Gaussian image.* In this representation, each normal vector is weighted by the surface area of the corresponding facet. Little [11] gives an iterative method for reconstructing convex objects, given their extended Gaussian image. In general, non-convex objects cannot be uniquely reconstructed.

Other important surface representations exploit a duality between points and planes in three dimensions. Duality is an important concept in geometry [1,7]. Dual space was originally proposed by Huffman as an aid in analyzing pictures of impossible objects [9] and later applied to interpreting general line drawings of polyhedral scenes [12,10]. Huffman's version of duality involves associating the plane

$$ax + by + cz + d = 0 \tag{1}$$

with the point $(-a, -b, -d)$ in dual space. In addition to the duality between points and planes, there is also an induced duality between lines in (x, y, z)-space and lines in the dual space. The dual of the line formed by the intersection of two planes is the line passing through the two points that are the duals of the planes. In Huffman's duality, only the first two coordinates of the dual point of a plane are related to orientation. *Gradient space,* another duality representation, is formed by orthogonally projecting the dual points (a, b, d) onto the plane $d = 1$. Shafer [21] provides extensive analysis describing the advantages and uses of duality and the gradient space in analyzing images for computer vision. Unfortunately, the interesting relationships between lines and points in the image and the dual lines and points are achieved under the assumption that the images are produced by orthogonal projection.

The *spherical dual representation (SDR)* dualizes planes into points by normalizing the constant d to -1 in Equation 1. Thus the plane

$$ax + by + cz - d = 0 \qquad d \neq 0$$

is mapped to the dual point $(a/d, b/d, c/d)$ in spherical dual space [19]. We name this dual the spherical dual since this normalization has spherical symmetry about the origin as opposed to the cylindrical symmetry of Huffman's duality. To represent a polyhedron, each facet is taken to be the point dual to the plane containing the facet. The dual point is the node of a graph called the SDR of the polyhedron. The node f corresponding to facet F is connected via an arc to the node f' corresponding to facet F' if facets F and F' share an edge. It is possible for two facets to share more than a single edge. The spherical dual representation does not explicitly represent such multiple adjacency. To accommodate multiple facets in the same plane, the spherical dual representation represents each facet as a different node in the graph; that is, two nodes carry the same label (planar equation) if the corresponding facets lie in the same plane. Henceforth, we identify each node in the SDR with its associated facet so that we can speak of a facet as *being* a node of the SDR. Figure 1 shows an object and its SDR (minus the planar equations). In view of the *graph* nature of the SDR, graph theoretic terms and operations apply to SDR. In fact, the spherical dual representation is the facet connectivity graph of the polyhedron, where each node has an attached planar equation. In contrast to the vertex connectivity graph, the SDR is always a connected graph as long as the surface of the polyhedron is connected.

Features of an object are high level abstractions that humans generally identify and operate with. Some examples of object features in manufacture, design, and recognition are boss, rib, blind hole, and through hole. Feature extraction at this abstract level is thus important in object recognition and geometric modeling systems. These features are further abstracted into *projecting features* and *depressions.* Falcidieno and Giannini [3] present a method for the automatic recognition and representation of shape-based features in a geometric modeling system. Loops in a face are the primary elements of this approach. The algorithm however requires the specification of the object as a face adjacency hypergraph, i.e., each loop on the face is explicitly specified. The algorithm we present also extracts the shape features in terms of projecting features and depressions. Our method has the advantage that the specification of the object is only as a face adjacency graph rather than the face adjacency hypergraph. The loop information is extracted automatically. However, the domain of our algorithm is currently restricted to genus 0 objects only. The ability to extract high level features automatically makes our representation very powerful in computer

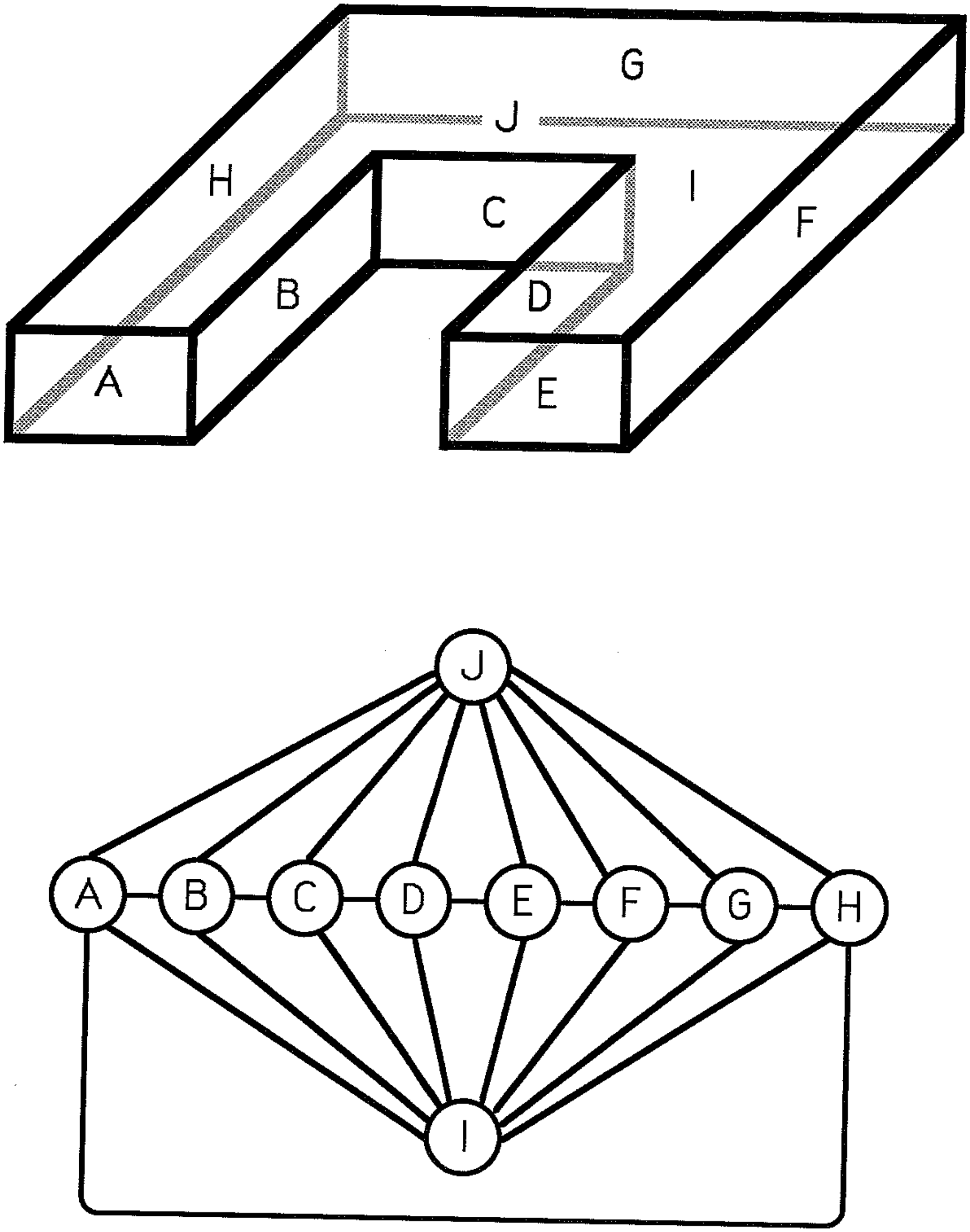

Figure 1: An object and its SDR

aided manufacture and computer vision applications.

3.3 Embeddings and Their Duals

Edmonds' theorem [2,4] states that every rotation of a graph G induces (up to orientation preserving equivalence of embeddings) a unique embedding of G into an oriented surface, i.e., there is a one-to-one correspondence between the possible orderings of the neighbors of each node and the embedding of the graph on some orientable surface. If a graph G has k nodes of respective degrees $d_1, d_2, \ldots, d_k$, then the total number of orientable embeddings is

$$\prod_{i=1}^{k} (d_i - 1)!.$$

Thus, in general, a graph does not have a unique embedding.

Given a connected graph G, a closed surface S, and an embedding $i : G \rightarrow S$, a *dual graph* G^* and a *dual embedding* $i^* : G^* \rightarrow S$ are defined as follows. For each region f of the embedding $i : G \rightarrow S$, place a node f^* in its interior. Then, for each arc e of the graph G, draw an arc e^* between the nodes just placed in the interiors of the regions containing e. The resulting graph with nodes f^* and arcs e^* is called the *dual graph* G^* for the embedding $i : G \rightarrow S$. The resulting embedding of the graph G^* in the surface S is called the *dual embedding.* Whitney shows that a triconnected planar graph has a *unique* planar embedding and hence a unique dual [23].

3.4 Reconstruction techniques

Hanrahan [6] gives a linear time algorithm for the unique reconstruction of a genus 0 polyhedron given its wire frame representation. This algorithm, however, requires the wire frame input of the polyhedron to be three-connected and planar in the graph theoretic sense. The facets of the polyhedron correspond to the faces in the unique planar embedding of the vertex connectivity graph.

Markowsky and Wesley [15] present an algorithm that generates all polyhedra with a given wire frame. This explicitly uses topological and geometric information by forcing the final facets to be planar. Human intervention is required to choose one of the several polyhedra reconstructed from such a wire frame representation.

Weiler [22] enumerates the boundary representations of polyhedra that are sufficient for unique reconstruction. Making use of Edmonds' theorem [2], Weiler shows that knowing the *ordered* set of edges around each vertex, or each edge, or each facet of a polyhedron is sufficient information for the reconstruction of any polyhedron. Weiler also states that a representation without order has insufficient information for unique reconstruction. Later, we show unique reconstructibility for genus 0 polyhedra when represented by SDR, a representation without order information. We also prove that a polyhedron of any genus that is represented by SDR and that only has facets with at most 4 edges is uniquely reconstructible.

A different approach to reconstruction of convex polyhedra is suggested by Minkowski's theorem [5]. Minkowski uniquely characterizes, up to a translation, any convex polyhedron by the *area* of its facets and their orientations. Using Minkowski and Brunn-Minkowski theorems [5], Little solves the problem of reconstructing the polytope given its extended Gaussian image by solving a constrained minimization problem [11]. The domain of Little's algorithm is the same as that of Minkowski's theorem; it fails to reconstruct non-convex polyhedra.

The most important result on the realization of a polyhedron from its vertex connectivity graph is Steinitz's theorem [5]. The theorem states that *a graph G is realizable as a convex polyhedron if and only if G is planar and triconnected.* In the case of reconstruction of polyhedra, this theorem can be used for all combinatorially convex polyhedra. A polyhedron is *combinatorially convex* if its vertex connectivity graph is planar and triconnected and the polyhedron has genus 0. From Whitney's theorem [23], every combinatorially convex polyhedron is uniquely reconstructible from its facet connectivity graph; i.e., its SDR. In the next section, we extend unique reconstructibility to *every* genus 0 polyhedron.

4 Reconstruction of Genus 0 Polyhedra

In this section, we present an algorithm RECONSTRUCT that uniquely reconstructs any genus 0 polyhedron P from its spherical dual representation. RECONSTRUCT first builds a graph for each facet in P and then extracts the vertices of each boundary of the facet from that graph.

Let $SDR = (N, A)$ be the spherical dual representation of the genus 0 polyhedron P. Let $\mathcal{P}(f)$ be the plane containing the facet $f \in N$. Each facet f of P consists of a bounded, connected region in $\mathcal{P}(f)$ that has one or more cycles of edges and vertices of P as boundary. If f is bounded by t cycles, then f has exactly $t-1$ holes. To reconstruct P, it suffices to determine all the bounding cycles of all facets. Let $\mathcal{F}(f)$ be the set of facets that are adjacent to f in SDR. If $f^* \in \mathcal{F}(f)$, then $\mathcal{L}(f, f^*) = \mathcal{P}(f) \cap \mathcal{P}(f^*)$ is an infinite line within $\mathcal{P}(f)$ that contains the (one or more) edges of P that are shared by f and f^*.

Suppose that $f^*, f^{**} \in \mathcal{F}(f)$ have the property that $\mathcal{L}(f, f^*)$ and $\mathcal{L}(f, f^{**})$ are not parallel. Then $\mathcal{L}(f, f^*)$ and $\mathcal{L}(f, f^{**})$ intersect at a point $\mathcal{T}(f, f^*, f^{**})$ within $\mathcal{P}(f)$. Let $\mathcal{Q}(f)$ be the set of all such intersections within $\mathcal{P}(f)$. Then every vertex v of P that is incident to f is an element of $\mathcal{Q}(f)$. In general, $\mathcal{Q}(f)$ contains many points that are not vertices of P. A necessary condition for a point $p \in \mathcal{Q}(f)$ to be a vertex of P is that there exist a *defining cycle* $f, f_1, \ldots, f_k$ in SDR such that $p \in \mathcal{Q}(f_i)$, $i = 1, \ldots, k$. If p is indeed a vertex of P, then there is, of course, a defining cycle for p which is the cycle of facets incident to p. However, a given p may have many defining cycles. That the existence of a defining cycle is not sufficient for p to be a vertex is shown by the example in Figure 2. This example is a truncated tetrahedron, where the facet E has cut off the top vertex p of the original tetrahedron. The point p is not a vertex of the truncated tetrahedron, yet $p \in \mathcal{Q}(A)$, $p \in \mathcal{Q}(B)$, $p \in \mathcal{Q}(C)$, and A, B, C is a cycle in SDR.

If a point $p \in \mathcal{Q}(f)$ meets the above necessary condition (of having a defining cycle), call p a *near-vertex.* The minimal subgraph $\mathcal{NF}(f)$ of SDR that contains all defining cycles of every near-vertex of f is the *near-facet graph* of f. Clearly, every node in $\mathcal{F}(f)$ is also adjacent to f in $\mathcal{NF}(f)$. Shortly, we will be embedding subgraphs of $\mathcal{NF}(f)$ in the plane and reading off the vertices incident to f from the faces of the embeddings. Any node of degree two in $\mathcal{NF}(f)$ has no effect on these embeddings and can be eliminated by *series reduction* (replace the node and its two incident arcs by a single arc; see Gross and Tucker [4]). If any parallel arcs are introduced by series reduction, all but one can be eliminated by *parallel reduction.* The *facet graph* $SDR(f)$ of f is $\mathcal{NF}(f)$ that has been reduced as much as possible by series and parallel reductions.

Since P has genus 0, $SDR(f)$ gives us all the information necessary to determine the bounding

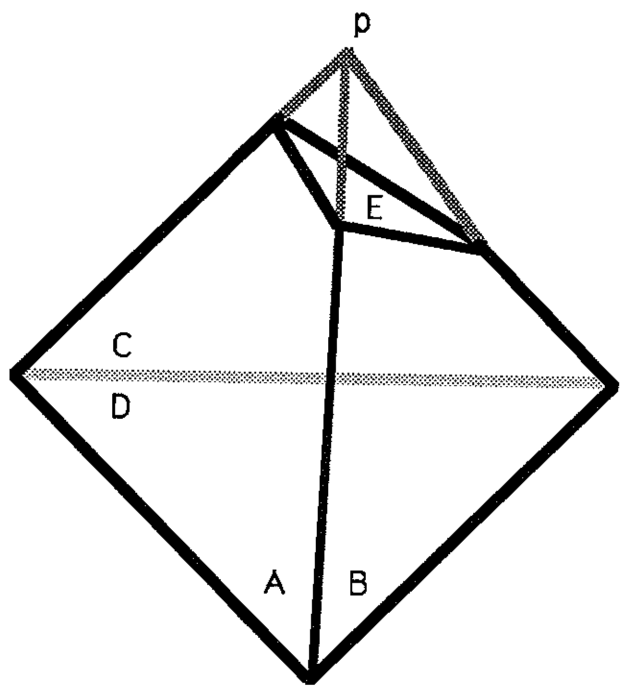

Figure 2: Truncated tetrahedron

cycles of f. For example, the number of bounding cycles is just the number of biconnected components of $SDR(f)$. This is illustrated in Figure 3 where facet F has two bounding cycles, and $SDR(F)$ has two biconnected components. Observe also that F is the sole articulation point of $SDR(F)$. This observation is formalized in the following theorem.

Theorem 1 *Let $SDR = (N, A)$ be the spherical dual representation of a genus 0 polyhedron P, and let f be a node of SDR. If $SDR(f)$ contains an articulation point, then f is the only articulation point. $SDR(f) - f$ has t connected components if and only if f has t bounding cycles.*

Proof: By the definition of $SDR(f)$, every $f^* \in \mathcal{F}(f) - \{f\}$ has two vertex disjoint paths to f in $SDR(f)$. Therefore, only f can be an articulation point of $SDR(f)$.

Assume that f has t bounding cycles. Define an equivalence relation $\equiv$ on $N - \{f\}$ such that $f^* \equiv f^{**}$ if there exists a curve on (the surface of) P that goes from a point in the interior of f^* to a point in the interior of f^{**} without passing through the closure of f. Because P has genus 0, $\equiv$ has

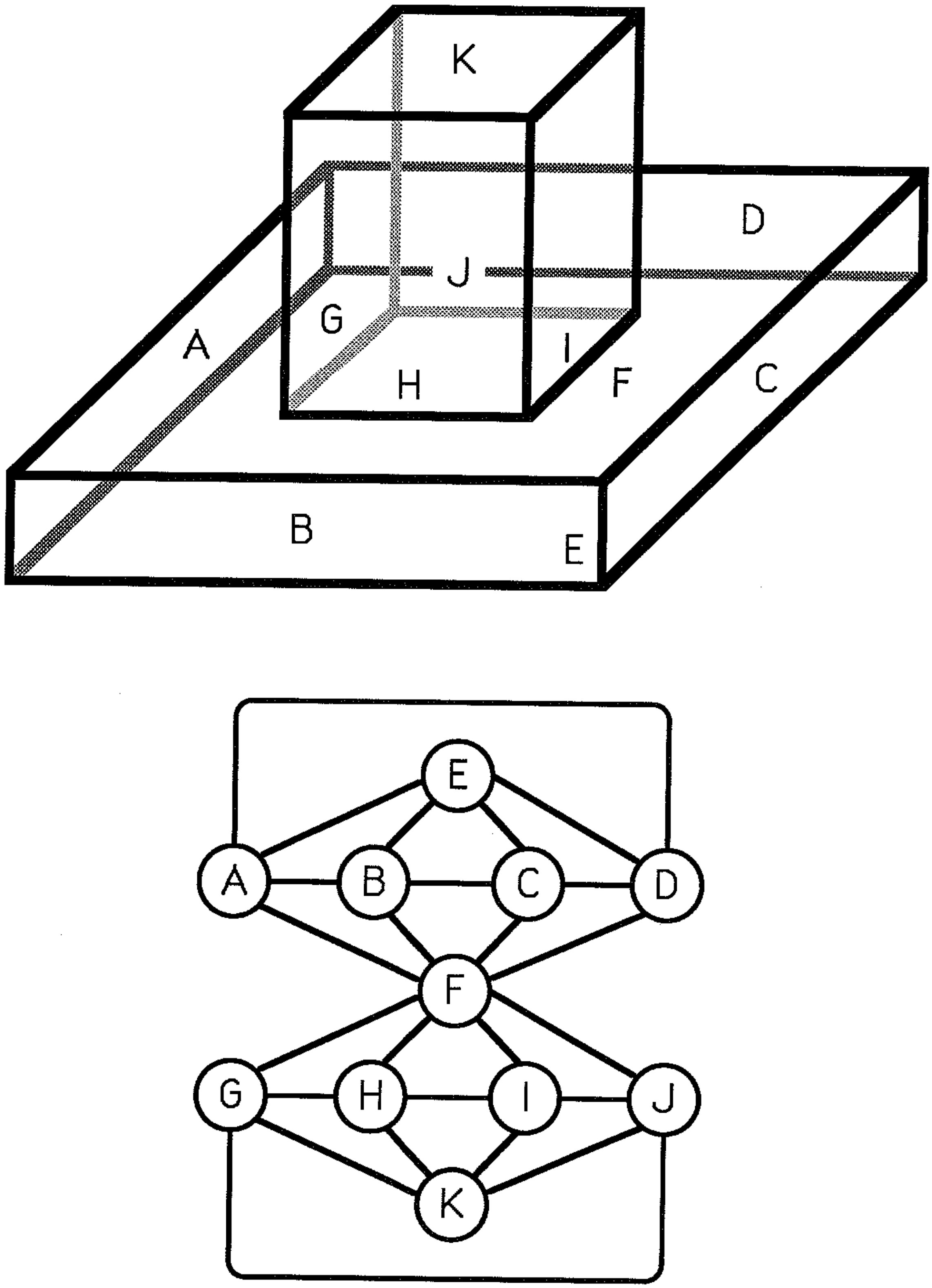

Figure 3: A polyhedron with an articulation point in its SDR

exactly t equivalence classes. $\mathcal{NF}(f) - f$ has one component for each equivalence class. It is easy to see that series and parallel reduction applies independently to each component of $\mathcal{NF}(f) - f$. Thus, $SDR(f) - f$ has the same number of components as $\mathcal{NF}(f) - f$, namely t. □

As SDR is planar but not necessarily triconnected, SDR does not have, in general, a unique embedding in the plane, whose dual would be the vertex connectivity graph of P. A first approach that is doomed to failure is to decompose SDR into its triconnected components, embed each in the plane, and somehow read off the structure of P from these embeddings. The failure of this approach is illustrated by the polyhedron in Figure 4, shown with its SDR. The triconnected components of SDR are shown embedded in the plane in Figure 5. There is no face in any of the embeddings that corresponds to the vertex of P shared by the facets F, C, I, and H, nor to the vertex shared by the facets F, C, I, and J. However, there is a "false" vertex indicated by the face bounded by the cycle of facets F, H, I, J.

In view of this failure, we turn to facet graphs for a solution. From the proof of Theorem 1, we know that if $SDR(f)$ has biconnected components $C_1, C_2, \ldots, C_t$, then each C_i contains f and corresponds precisely to one of the bounding cycles of f. As each C_i may be processed separately to determine its corresponding bounding cycle, we henceforth assume that $SDR(f)$ contains only one biconnected component, namely $SDR(f)$ itself.

It is now our task to determine the vertices that occur on the bounding cycle of f and the order in which they occur. If $SDR(f)$ is triconnected, then it has a unique planar embedding. Suppose that the order of the nodes in $\mathcal{F}(f)$ about f in this embedding is $f_1, f_2, \ldots, f_k$. These correspond to the vertices

$$\mathcal{L}(f, f_1, f_2), \mathcal{L}(f, f_2, f_3), \ldots, \mathcal{L}(f, f_k, f_1),$$

in that order, defining the bounding cycle of f. Call this cycle of vertices the *cycle induced by the embedding.*

If $SDR(f)$ is not triconnected, then it may not be true that all embeddings induce the same cycle of vertices or even that there exists an embedding that induces a cycle equal to the bounding cycle of f. We note that by the definition of $SDR(f)$:

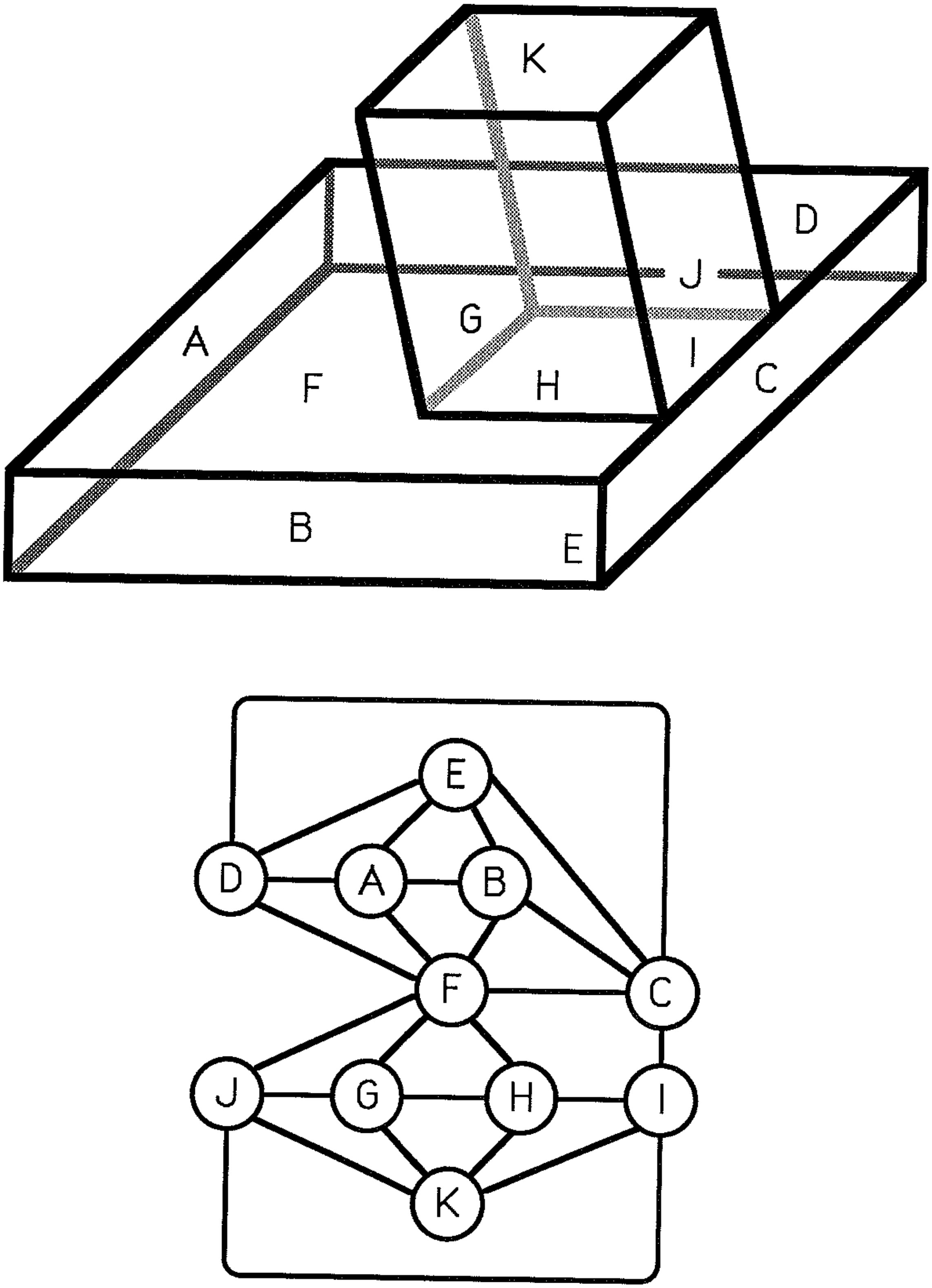

Figure 4: A polyhedron whose SDR is not triconnected

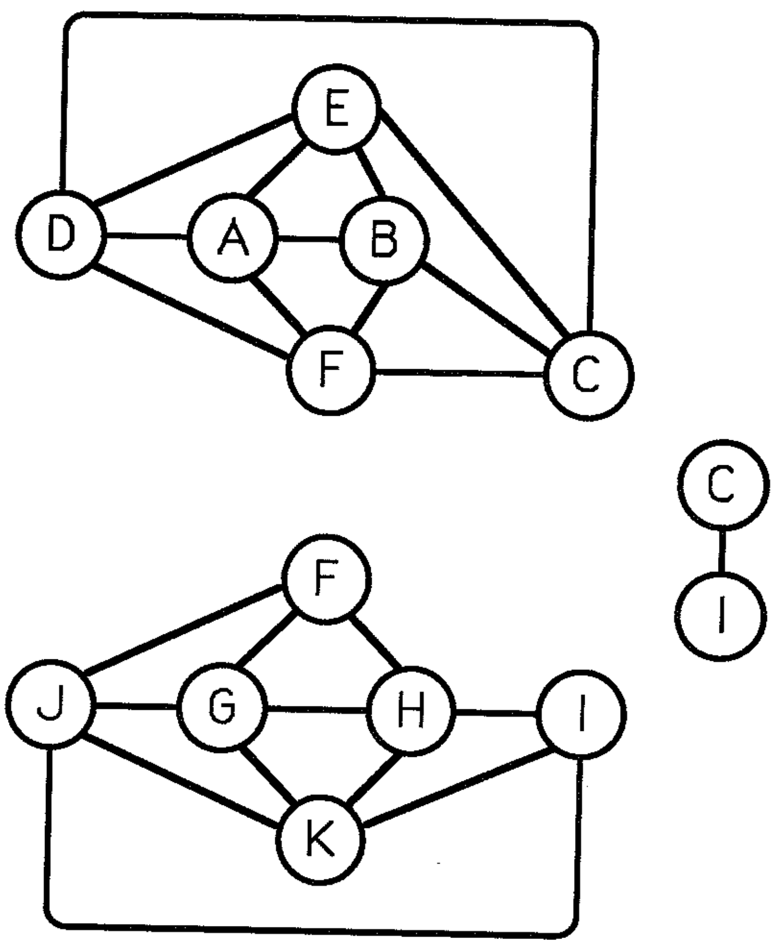

Figure 5: The corresponding triconnected components

Proposition 2 *If $SDR(f)$ is biconnected, then any separation pair of $SDR(f)$ contains f.*

Thus any separation pair of $SDR(f)$ has the form (f, f^*), where f^* is a node of $SDR(f)$ which may or may not be adjacent to f. Call f^* a *separation partner* of f. All separation partners of f can be identified by finding the articulation points of $SDR(f) - f$. For example, Figure 6 shows the facet graph of the facet F in the polyhedron of Figure 4. The separation partners of f are facets C and I. Note that F and C are adjacent in $SDR(f)$, while F and I are not.

The example in Figure 4 is a degenerate one in that $\mathcal{P}(F)$, $\mathcal{P}(C)$, and $\mathcal{P}(I)$ intersect in a single line. Figure 7 shows a non-degenerate example in which there is a single separation pair whose

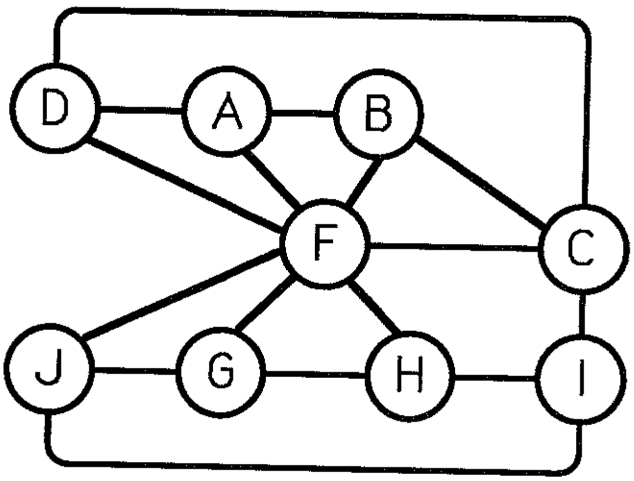

Figure 6: The facet graph $SDR(F)$

nodes are not adjacent. The polyhedron is a box with a raised pyramid (facets C, D, E, and F) on its front face. Facets A and B are distinct facets that reside in the same plane and that are not adjacent in SDR. Facet B is a separation partner of A (and vice versa). Note that A and B share (are incident to) both vertices v_1 and v_2. This is a general phenomenon.

Proposition 3 *If f^* is a separation partner of f and $f^* \notin \mathcal{F}(f)$, then f and f^* have two or more shared vertices but no shared edges.*

We can make a stronger observation. Let f^* be a separation partner of f. Suppose SDR^* is a connected component of $SDR(f) - f - f^*$. Then there exists a closed curve contained in the closure (in the topological sense) of $f \cup f^*$ that separates P into two regions, each homeomorphic to a disk, such that SDR^* is contained wholly in one of the regions and the remainder of $SDR(f) - f - f^*$ is contained wholly in the other region.

Note that SDR^* may or may not contain a separation partner of f. If it does not, then we can determine the boundary between f and SDR^* as follows. Construct the *decomposition graph* $SDR^*(f^*)$ by taking the subgraph of $SDR(f)$ induced on f, f^*, and the nodes of SDR^*, and add the arc (f, f^*) if $f^* \notin \mathcal{F}(f)$. The following observation is key.

Proposition 4 *If SDR^* contains no separation partner of f, then $SDR^*(f^*)$ is triconnected.*

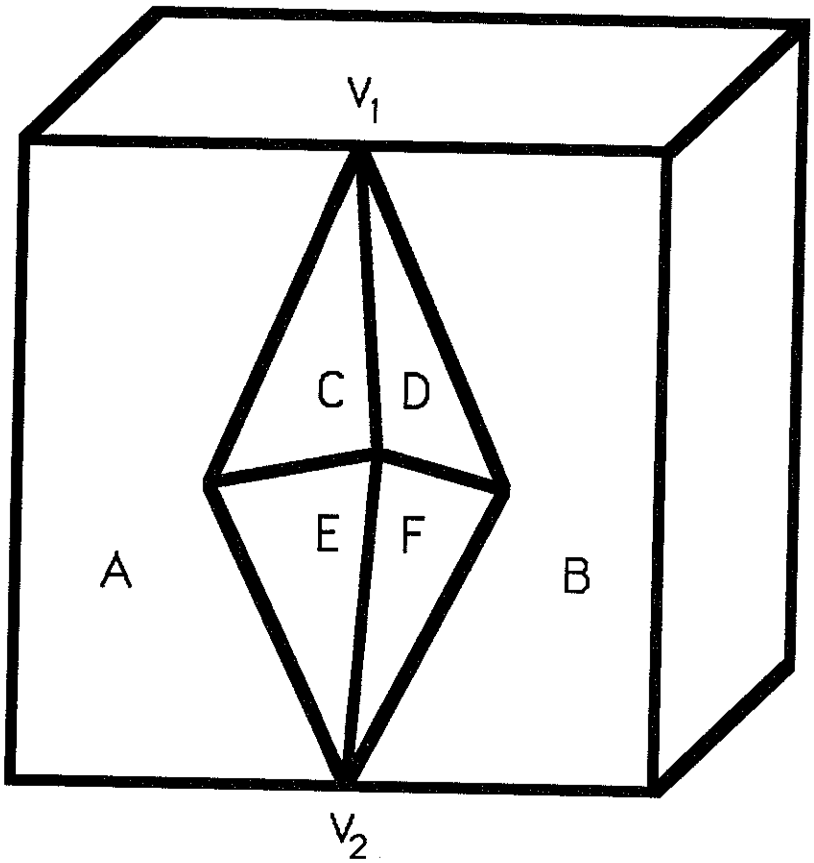

Figure 7: A separation pair whose nodes are not adjacent

Hence, $SDR^*(f^*)$ has a unique planar embedding. The order of arcs in $\mathcal{F}(f)$ around f in this embedding exactly gives the order in which facets of $SDR^*(f^*)$ appear in the boundary between f and SDR^*. As is true when $SDR(f)$ is triconnected, the cycle induced by the embedding gives the (cyclic) order of vertices and edges that form the bounding path. The cycle breaks into a path at the arc (f, f^*).

For example, if we apply this decomposition to the facet graph $SDR(F)$ in Figure 6, we obtain two decomposition graphs as shown in Figure 8. The arc (F, I) has been added to the lower decomposition graph. From the two planar embeddings, we learn that the bounding cycle of F has facets A, B, C, and D in that order in one path and facets G, H, and J in that order in a second path. The paths begin and end at the separation partners of F. Therefore, the second path

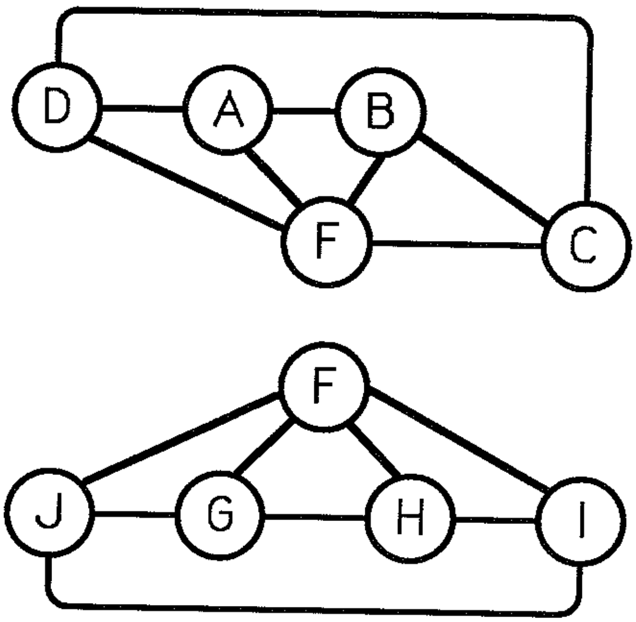

Figure 8: The decomposition graphs of $SDR(F)$

breaks between H and J (since $I \notin \mathcal{F}(F)$), and the end vertices of the path occur at $\mathcal{T}(F, H, I)$ and $\mathcal{T}(F, J, I)$. Both of these points are on the line $\mathcal{L}(F, C)$, so clearly these are the vertices at which the first and second path connect.

For any $SDR(f)$ that is not triconnected, there always exists an SDR^* that contains no separation partner for f (examine the tree of biconnected components and articulation points of $SDR(f) - f$ to find a biconnected component that is a leaf). We can then determine the subpath of the bounding cycle of f that is between f and SDR^* by the above decomposition method. Once the subpath is determined, we would like to remove SDR^* from further consideration. This can be done by replacing $SDR(f)$ by the *reduced graph* $SDR(f) - SDR^*$ with the arc (f, f^*) added, if $f^* \notin \mathcal{F}(f)$. In geometric terms, this reduction amounts to removing the feature of P that corresponds to SDR^* and replacing it by an edge in P shared by f and f^*. Geometrically this may not always work, as witnessed by the polyhedron in Figure 7, where facets A and B are in the same plane and therefore cannot share an edge. However, combinatorially the reduction does work.

The strategy for finding the bounding cycle of $SDR(f)$ now is clear. Iteratively find a decomposition graph $SDR^*(f^*)$ that contains no separation partner of f, determine the corresponding bounding path, and reduce $SDR(f)$. Once $SDR(f)$ is reduced to a triconnected graph, construct the bounding cycle of f by gluing the subpaths together at the separation partners of f. This completes the description of the processing of each biconnected component of $SDR(f)$.

In summary, the algorithm RECONSTRUCT consists of the following steps, applied to each facet f.

1. Form the set $Q(f)$.
2. Determine $\mathcal{NF}(f)$ and reduce it to $SDR(f)$.
3. Decompose $SDR(f)$ into its biconnected components, $SDR_1, SDR_2, \ldots, SDR_k$, $k \geq 1$.
4. For each biconnected component SDR_i, determine the bounding cycle of f corresponding to SDR_i, using the decomposition graph strategy.

As the size of $SDR(f)$ may be $\Theta(|N|)$, the time complexity of these steps for each facet is $O(|N|)$. The total time complexity for RECONSTRUCT is $O(|N|^2)$, though we expect that it is typically much less.

Theorem 5 *Algorithm* RECONSTRUCT *uniquely reconstructs any genus* 0 *polyhedron given its* SDR*. The time complexity of* RECONSTRUCT *is* $O(n^2)$*, where* $n = |N|$ *is the size of the* SDR*.*

Algorithm RECONSTRUCT successfully reconstructs some, but not all, polyhedra of genus greater than zero. If P is a polyhedron of arbitrary genus, then it is possible that, for some facet f, $SDR(f)$ is not even planar. This occurs when (one or more) cycles in $SDR(f)$ pass through (one or more) holes in P. Also, a facet f with multiple bounding cycles need not even be an articulation point in $SDR(f)$ if a hole of P passes through f.

RECONSTRUCT can be modified to successfully reconstruct more higher-genus polyhedra as follows. Typically, the facets in some non-empty subset of N can be successfully reconstructed by the steps in RECONSTRUCT. Boundary information from these reconstructed facets can be shared with neighboring facets that are not immediately reconstructible, perhaps making them

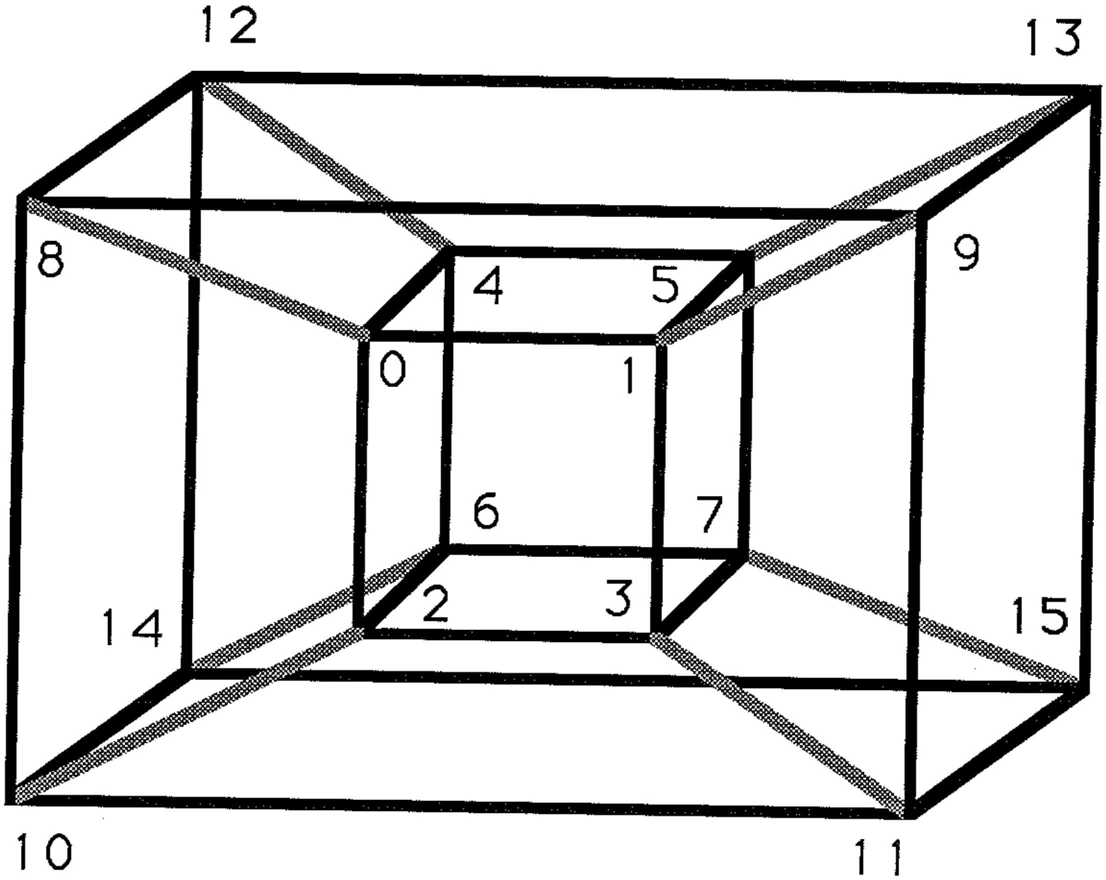

Figure 9: The hypercube connectivity graph

reconstructible in the process. If such information sharing propagates to all facets of P, then the modified RECONSTRUCT successfully reconstructs P. We have found that this modified algorithm is capable fo reconstructing a number of "hard" polyhedra that we had proposed as potential counterexamples to unique reconstructibility for polyhedra of higher genus.

The SDR shown in Figure 9 is a graph that cannot be solved by the above (modified) algorithm. The genus of this graph is 1. The $SDR(f)$ of each facet of the polyhedron is non-planar and triconnected (each $SDR(f)$ is homeomorphic to K_5). Hence not even a single face can be reconstructed via planar embedding of its $SDR(f)$. In the next section, however, we show that the graph in Figure 9 does have a unique reconstruction.

5 The Hypercube Is Uniquely Reconstructible

In this section, we show that the hypercube, a classic example of the ambiguity of the wire frame representation scheme, is unambiguous when the same graph is viewed as the SDR of a polyhedron.

The hypercube is the connectivity structure of the graph in Figure 9. It is a classic example of the ambiguity of the vertex connectivity graph representation (wire frame model) [14]. From Figure 9, it is clear that three different genus 1 polyhedra have this as a vertex connectivity graph, one polyhedron for each direction that the hole can take. For example, one realization has a hole through the vertex cycles represented by $0, 1, 5, 4$ and $2, 3, 7, 6$. The dual graph of each of the three realizations of the graph in Figure 9 is again a hypercube. In contrast to the ambiguity of the vertex connectivity representation, we show that the hypercube as an SDR is unambiguous.

Each facet of a polyhedron having the hypercube as SDR is connected by an arc to four other facets. The four lines defined by the facet adjacencies form at most two different quadrilaterals. (If two of the lines are parallel, only one quadrilateral is formed, and there is no ambiguity.) Refer to Figure 10 which depicts the plane containing facet 1 and the four lines formed by intersection with the planes containing facets 0, 3, 5, and 9. There are two possible interpretations for the boundary of each facet. One interpretation has vertices a, f, b, and e, while the other interpretation has vertices a, d, b, and c. Two vertices, a and b, are present in both interpretations. Vertex a, called the *fixed vertex,* has its context unchanged, i.e., when we follow the boundary of the two polygons in the same direction, the line segments occur in the same order. Vertex b, called the *reflex vertex,* has its context reversed. The internal angle at the reflex vertex changes from being a convex angle ($\angle dbc$) in one interpretation to concave ($\angle fbe$) in the other. It is immediately clear from Figure 10 that fixing any one of the remaining four vertices determines the polygon unambiguously. These four vertices are termed the *transient* vertices. Vertices on each of the lines through the fixed vertex and nearer the fixed vertex are called *intruded vertices* and those farther away are called *extruded vertices.* Thus in facet 1 in Figure 10, vertices c and d are intruded vertices and e and f are extruded vertices.

Figure 10 also shows the resulting facet subgraph for each of the two interpretations. The quadrilateral $afbe$ corresponds to the subgraph on the left, while the quadrilateral $adbc$ corresponds

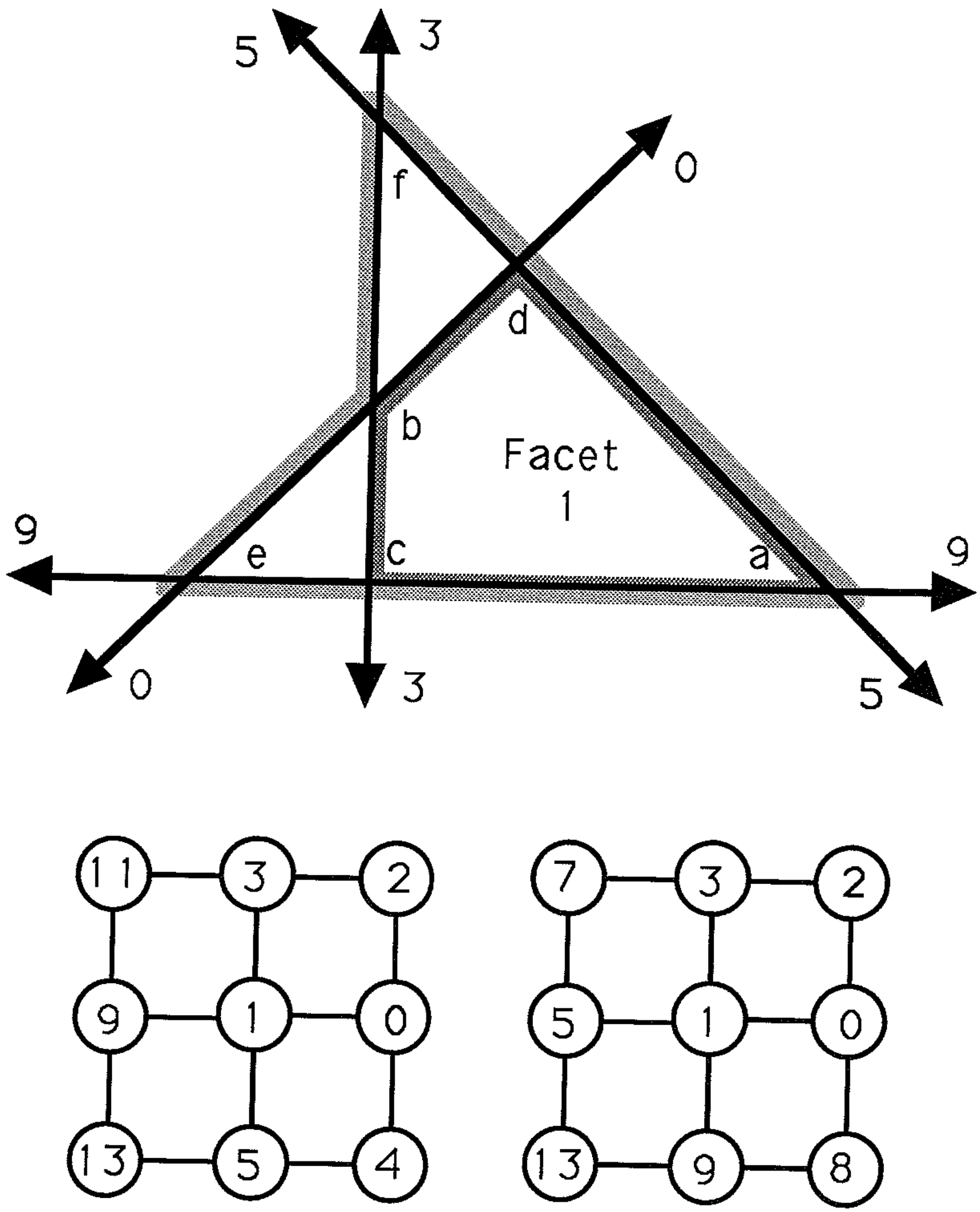

Figure 10: Two realizations of facet 1 and resulting subgraphs

to the subgraph on the right. Observe that the transient vertices are either both intruded or both extruded. Observe also that the angle at the reflex vertex is less than 180° if the intruded vertices are chosen and is greater than 180° if the extruded vertices are chosen. Suppose a particular choice of the quadrilaterals on facet 1 is made. This completely determines the quadrilaterals of the four facets adjacent to it. Thus, once a particular choice at one facet has been made, the entire hypercube is determined.

Since in a polyhedron no two adjacent facets can both have angles greater than 180° at a common vertex, it is impossible to have ambiguity when one interpretation has all convex internal angles at the reflex vertex since in the other interpretation all the angles at the reflex vertex must be concave. By the same reasoning, the only possible way for there to be two interpretations at the reflex vertex is to have alternating convex and concave angles at the reflex angle on each of the four facets, i.e., an intruded vertex on a facet i is an extruded vertex on a facet j if i is adjacent to j, and i and j share a reflex vertex. For example, the vertex defined by facets 1, 5, 8 and 4 is intruded on facet 1 and is extruded on facet 4. A labeling of the four facets forming the reflex vertex is given in Figure 11. The fixed vertices are labeled p_i, $i = 1, \dots, 4$, the reflex vertex is labeled O, and the transient vertices are labeled q_i, $i = 1, \dots, 8$.

Lemma 6 *For every reflex vertex r of the hypercube, there is only one interpretation for each of its four facets.*

Proof: To obtain a contradiction, assume that there is a reflex vertex r with two interpretations. Without loss of generality, let the reflex vertex be $r = (0, 0, 0)$ (labeled O for origin in Figure 11).

The following linear constraint equations can be obtained from Figure 11. (Note that these are 3-dimensional vector equations.)

$$t_1(q_8 - p_1) = q_2 - p_1 \tag{2}$$

$$t_2(q_1 - p_1) = q_7 - p_1 \tag{3}$$

$$t_3(q_2 - p_2) = q_4 - p_2 \tag{4}$$

$$t_4(q_3 - p_2) = q_1 - p_2 \tag{5}$$

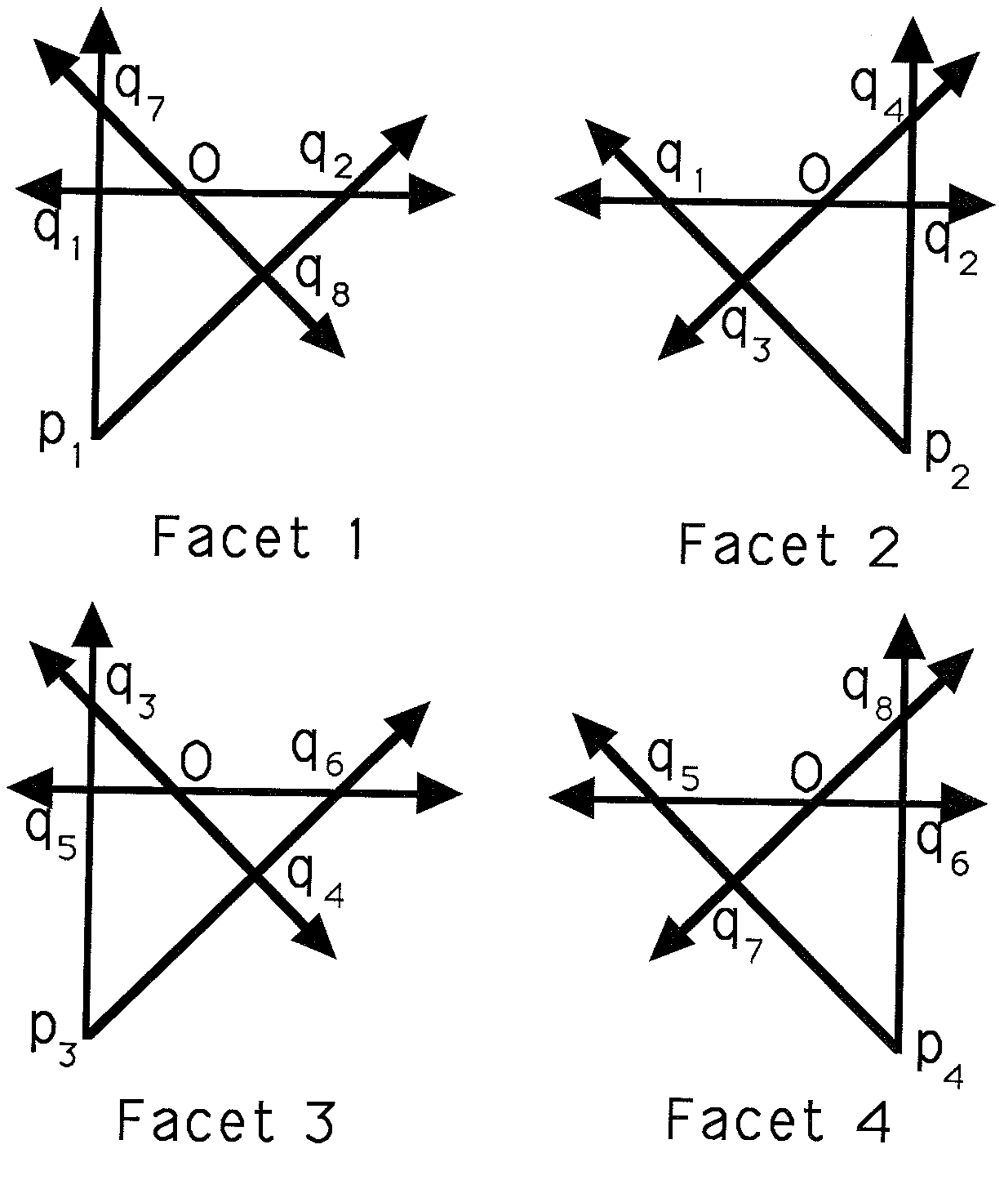

Figure 11: Four facets at a reflex vertex.

$$
\begin{aligned}
t_5(q_4 - p_3) &= q_6 - p_3 && (6)\\
t_6(q_5 - p_3) &= q_3 - p_3 && (7)\\
t_7(q_6 - p_4) &= q_8 - p_4 && (8)\\
t_8(q_7 - p_4) &= q_5 - p_4 && (9)\\
t_9 \cdot q_1 &= q_2 && (10)\\
t_{10} \cdot q_3 &= q_4 && (11)\\
t_{11} \cdot q_5 &= q_6 && (12)\\
t_{12} \cdot q_7 &= q_8 && (13)
\end{aligned}
$$

where for $1 \le i \le 8$, $t_i > 1$, and for $9 \le i \le 12$, $t_i < 0$. Eliminating p_1 using Equations 2 and 3, p_2 using Equations 4 and 5, p_3 using Equations 6 and 7, and p_4 using Equations 8 and 9, we obtain

$$
\begin{aligned}
\frac{(t_2 - 1)}{(t_1 - 1)}(t_1 q_8 - t_9 q_1) &= t_2 q_1 - q_7 && (14)\\
\frac{(t_4 - 1)}{(t_3 - 1)}(t_3 t_9 q_1 - q_4) &= t_4 q_3 - q_1 && (15)\\
\frac{(t_6 - 1)}{(t_5 - 1)}(t_5 q_4 - t_{11} q_5) &= t_6 q_5 - q_3 && (16)\\
\frac{(t_8 - 1)}{(t_7 - 1)}(t_7 t_{11} q_5 - q_8) &= t_8 q_7 - q_5 && (17)
\end{aligned}
$$

Substituting for q_4 and q_8 from Equations 11 and 13 into Equations 14, 15, 16, and 17 and simplifying, we obtain

$$
\begin{aligned}
(t_2(t_1 - 1) + (t_2 - 1)t_9)q_1 - ((t_1 - 1) + (t_2 - 1)t_1 t_{12})q_7 &= 0\\
(t_4(t_3 - 1) + (t_4 - 1)t_{10})q_3 - ((t_3 - 1) + (t_4 - 1)t_3 t_9)q_1 &= 0\\
(t_6(t_5 - 1) + (t_6 - 1)t_{11})q_5 - ((t_5 - 1) + (t_6 - 1)t_5 t_{10})q_3 &= 0\\
(t_8(t_7 - 1) + (t_8 - 1)t_{12})q_7 - ((t_7 - 1) + (t_8 - 1)t_7 t_{11})q_5 &= 0.
\end{aligned}
$$

Since no line passing through the pair of points appearing in any of the above 4 equations pass through the origin, the coefficient of each q_i in these equations is equal to zero. Thus we have

$$
\begin{aligned}
t_2(t_1 - 1) + (t_2 - 1)t_9 &= 0 && (18)\\
(t_1 - 1) + (t_2 - 1)t_1 t_{12} &= 0 && (19)
\end{aligned}
$$

$$t_4(t_3 - 1) + (t_4 - 1)t_{10} = 0 \tag{20}$$

$$(t_3 - 1) + (t_4 - 1)t_3 t_9 = 0 \tag{21}$$

$$t_6(t_5 - 1) + (t_6 - 1)t_{11} = 0 \tag{22}$$

$$(t_5 - 1) + (t_6 - 1)t_5 t_{10} = 0 \tag{23}$$

$$t_8(t_7 - 1) + (t_8 - 1)t_{12} = 0 \tag{24}$$

$$(t_7 - 1) + (t_8 - 1)t_7 t_{11} = 0 \tag{25}$$

Substituting for t_9, t_{10}, t_{11}, and t_{12} from Equations 18, 20, 22, and 24 into Equations 21, 23, 25, and 19, respectively, gives

$$\begin{aligned}
\frac{(t_1 - 1)}{(t_2 - 1)} &= t_8 t_1 \frac{(t_7 - 1)}{(t_8 - 1)} \\
\frac{(t_3 - 1)}{(t_4 - 1)} &= t_2 t_3 \frac{(t_1 - 1)}{(t_2 - 1)} \\
\frac{(t_5 - 1)}{(t_6 - 1)} &= t_4 t_5 \frac{(t_3 - 1)}{(t_4 - 1)} \\
\frac{(t_7 - 1)}{(t_8 - 1)} &= t_6 t_7 \frac{(t_5 - 1)}{(t_6 - 1)}
\end{aligned}$$

which after further algebraic simplification yields

$$\frac{(t_1 - 1)}{(t_2 - 1)} = t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 \frac{(t_1 - 1)}{(t_2 - 1)}$$

and finally

$$t_1 t_2 t_3 t_4 t_5 t_6 t_7 t_8 = 1.$$

This is a contradiction, since $t_i > 1,\ i = 1, \ldots, 8$. The lemma follows. □

Since only one facet is sufficient to determine the polyhedron unambiguously, the hypercube, viewed as an SDR represents a polyhedron unambiguously. The key result follows:

Theorem 7 *The hypercube is unambiguous as an SDR.*

6 Spherical dual graphs of degree 4 represent unique polyhedra

In this section, we generalize the result of the last section to the SDR of any polyhedron having degree ≤ 4. From the arguments in the previous section, it is clear that there cannot be any ambiguity in the interpretation of a quadrilateral formed by four arbitrary lines if any one of the transient vertices is known. Thus if any degree 4 facet f_i has a degree three facet adjacent to it, f_i is unambiguous. All degree 4 facets adjacent to f_i can then be resolved. Hence the SDR of any polyhedron of maximum degree 4 with at least one node of degree 3 represents a unique polyhedron.

Now we show that there cannot be ambiguity at a reflex vertex which has an odd number of facets incident on it. Since there is a switch from convex to concave angles between the two interpretations, ambiguity at a reflex vertex with an odd number of facets would imply that there are at least two adjacent facets each of which has an internal angle greater than $180°$. This is impossible in polyhedra. Thus there cannot be any ambiguity at a reflex vertex where an odd number of facets come together.

The remaining case to be considered is when an even number of facets come together at a reflex vertex, and all the nodes are of degree 4. We now prove that for this case also there is only one interpretation.

Lemma 8 *An SDR of degree* 4 *with an even number of facets forming a reflex vertex represents a unique polyhedron.*

Proof: Let f_i be a facet involved in forming a reflex vertex and n the degree of the reflex vertex. Without loss of generality, we may assume that the reflex vertex is at the origin. The fixed vertex on f_i is p_i, $i = 1, \ldots, n$, while the intruded vertices are q_{2i-2} and q_{2i-1}. The extruded vertices are q_{2i} and q_{2i-3}. The subscripts of the vertices may be incremented by $2n$ so as to fall in the range $1..2n$. As an illustration, if there are 6 facets meeting at a reflex vertex, and we are considering the vertex labels on facet 1, the intruded vertices are q_{12} and q_1, and the extruded vertices are q_2 and q_{11} respectively. The line through the odd numbered vertices q_{2i-3} and q_{2i-1} passes through p_i and is formed by the intersection of f_i with one adjacent facet. Similarly, the line through the

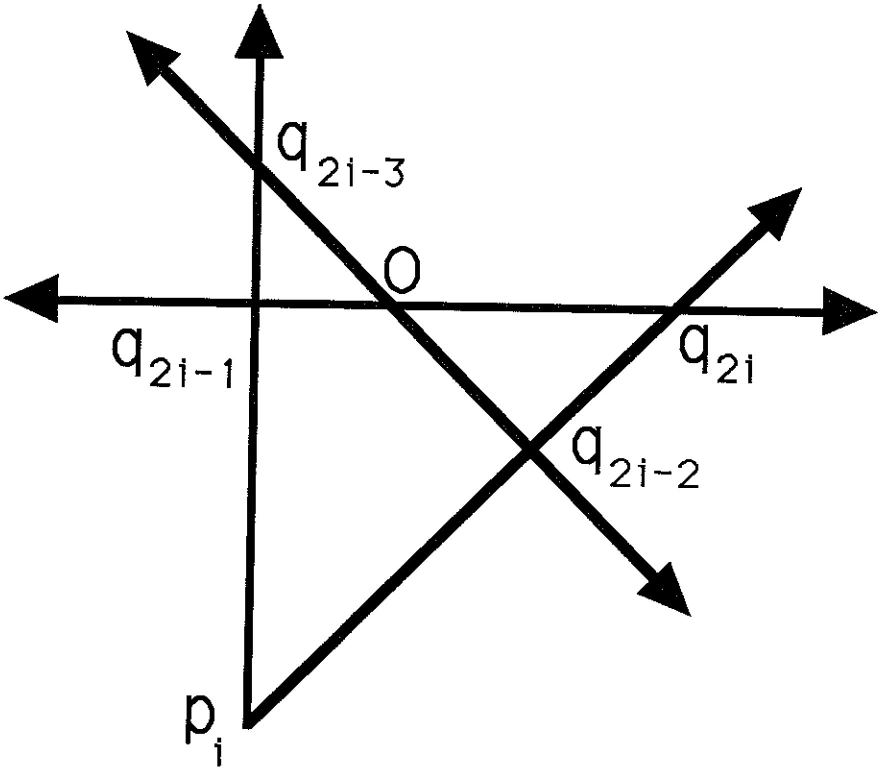

Figure 12: Vertex labeling of the realizations of facet f_i.

even numbered vertices q_{2i-2} and q_{2i} passes through p_i. See Figure 12. From the n facets meeting at the reflex vertex, the following relationships follow

$$t_{2i-1}(q_{2i-2} - p_i) = q_{2i} - p_i \tag{26}$$

$$t_{2i}(q_{2i-1} - p_i) = q_{2i-3} - p_i \tag{27}$$

$$u_i q_{2i-1} = q_{2i} \tag{28}$$

where $i = 1, \ldots, n$. Substitute q_{2i} for $i = 1, \ldots, n$, by $u_i q_{2i-1}$ from Equation 28 into Equations 26 and 27 obtaining

$$(t_{2i}(t_{2i-1} - 1) + (t_{2i} - 1)u_i)q_{2i-1} - ((t_{2i-1} - 1) + (t_{2i} - 1)t_{2i-1}u_{i-1})q_{2i-3} = 0. \tag{29}$$

From the geometry of the lines in Figure 12 and the fact that the origin is the reflex vertex, the coefficient of each point q_i in Equation 29 must be zero, i.e.,

$$t_{2i}(t_{2i-1}-1)+(t_{2i}-1)u_i = 0 \tag{30}$$

$$(t_{2i-1}-1)+(t_{2i}-1)t_{2i-1}u_{i-1} = 0. \tag{31}$$

Replace u_i, $i=1,\ldots,n$ in Equation 31 using Equation 30 obtaining

$$\frac{(t_{2i}-1)}{(t_{2i-1}-1)} = t_{2i}t_{2(i+1)-1}\frac{(t_{2(i+1)}-1)}{(t_{2(i+1)-1}-1)}$$

which yields

$$\frac{(t_2-1)}{(t_1-1)} = t_1t_2\ldots t_{2n}\frac{(t_2-1)}{(t_1-1)},$$

and finally,

$$t_1t_2t_3\ldots t_{2n}=1.$$

This is a contradiction, since $t_i>1$, $i=1,\ldots,2n$. The lemma follows. □

Thus there cannot exist an even number of facets forming a reflex vertex such that every facet has two interpretations. Since there is no ambiguity at the reflex vertex, all the facets of the polyhedron are determined unambiguously and we have the following theorem.

Theorem 9 *The SDR of polyhedra of any genus is* unambiguous *when the degree of the representation is at most 4.*

The proof given above also provides an algorithm for reconstruction. Ambiguity at one reflex vertex is resolved using the above equations. This can be done in time proportional to the number of facets incident to the reflex vertex. The rest of the polyhedron is then reconstructed in time proportional to the size of the polyhedron. Thus the time complexity of the algorithm is $O(n)$, where n is the size of the SDR.

7 Conclusion

Using the spherical dual representation, we have relaxed the requirement that the facet connectivity graph of a genus 0 polyhedron must be triconnected to support unique reconstruction. All genus 0 polyhedra are uniquely reconstructible. We have also extended unique reconstruction to any polyhedron of arbitrary genus, whose SDR has degree at most 4. The completeness of arbitrary spherical dual representations remains an open question. However, as numerous attempts to construct a polyhedron of higher genus that is a counterexample have failed, we are lead to the following conjecture:

Conjecture 1 *The spherical dual representation of a polyhedron of arbitrary genus is uniquely reconstructible.*

This conjecture is the most intriguing question raised by this work. A resolution to this conjecture will require a deep understanding of the structure of planar-faced polyhedra of arbitrary genus. As a taste of what such understanding should entail, we offer this smaller conjecture:

Conjecture 2 *No complete graph K_s, $s > 4$, is the spherical dual representation of any planar-faced polyhedron.*

We also note that there is a *global* version of RECONSTRUCT that also reconstructs genus 0 polyhedra. It must consider what happens when SDR is separated by a separation pair in the same manner as RECONSTRUCT. We do not elaborate on this version as it is less likely to lead to a universal reconstruction algorithm for polyhedra of arbitrary genus.

We have already mentioned in the description of RECONSTRUCT how that algorithm can naturally extract many features (protrusions and depressions) of a solid. The lack of explicit order information in our approach is an advantage over the approach to feature extraction taken by Falcidieno and Giannini [3].

An important observation is that RECONSTRUCT uses the requirement that facets are planar only to calculate $Q(f)$ and the edges of P. Therefore, RECONSTRUCT actually applies to larger classes of objects than we have allowed. It would be useful to specify and investigate other classes of "polyhedra" for which edges and the near-vertices in $Q(f)$ are easy to calculate.

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