Large Deformations of Rotating Polygonal Space Structures

By Layne T. Watson and C.Y. Wang

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Layne. T. Watson†
Virginia Polytechnic Institute & State University, Blacksburg, VA
C.-Y. Wang*
Michigan State University, East Lansing, MI

Introduction

The effects of rotation on space structures are very important nowadays. Because the structures are large and thus relatively flexible, unacceptable stresses or deformations may occur due to centrifugal forces. In a recent paper[1] we considered the effects of rotation on an unbalanced circular ring. The present note studies a related problem, i.e., the free rotation of polygonal frames. These shapes are basic in structural design.

Formulation

Figure 1a shows a polygonal frame of N sides, each side consisting of an originally straight elastic rod of length ℓ. The frame is rotated in its own plane about the center of gravity with angular velocity Ω. The rods may be rigidly joined, or they may be hinged together. In the latter case the polygonal shape is maintained only by rotation.

Let the origin of Cartesian axes (x', y') be at the center of mass and the x'-axis pass through one of the joints at point A (Fig. 1b). Let s' be the arc length along the frame from that point and θ be the local angle of inclination from the x'-axis. Due to symmetry the angle between the joints $2\pi/N$ is constant for all rotation rates. Thus we need to consider only the segment from $s' = 0$ to $s' = \ell/2$. Normalize all lengths by ℓ and drop the primes. The large deformation governing equation is[1]

$$\frac{d\theta}{ds}\frac{d^4\theta}{ds^4} - \frac{d^2\theta}{ds^2}\frac{d^3\theta}{ds^3} + \frac{d^2\theta}{ds^2}\left(\frac{d\theta}{ds}\right)^3 =$$
$$B\left[\frac{d^2\theta}{ds^2}(-y\cos\theta + x\sin\theta) - 2\left(\frac{d\theta}{ds}\right)^2 (y\sin\theta + x\cos\theta)\right]. \tag{1}$$

Here $B \equiv \rho\Omega^2\ell^4/(\text{flexural rigidity})$ is the important nondimensional free rotation parameter. The coordinates are related by

$$\frac{dx}{ds} = \cos\theta, \qquad \frac{dy}{ds} = \sin\theta. \tag{2}$$

Eq. (1) can be integrated twice to obtain

$$\frac{d^2\theta}{ds^2} = Y\cos\theta - X\sin\theta, \tag{3}$$

$$\frac{dY}{ds} = By, \qquad \frac{dX}{ds} = Bx. \tag{4}$$

† Professor, Department of Computer Science; supported in part by DOE Grant DE-FG05-88ER25068.

* Professor, Departments of Mathematics and Mechanical Engineering.

The boundary conditions are

$$x(0) = a, \qquad y(0) = 0, \tag{5}$$
$$X(0) = 0, \qquad Y(0) = -H, \tag{6}$$
$$y(1/2) = x(1/2)\tan(\pi/N), \qquad \theta(1/2) = \frac{\pi}{2} + \frac{\pi}{N}, \qquad \theta''(1/2) = 0, \tag{7}$$

plus either

$$\theta(0) = \frac{\pi}{2} + \frac{\pi}{N} \tag{8}$$

if the rods are rigidly joined, or

$$\theta'(0) = 0 \tag{9}$$

if the rods are hinged. There are eight boundary conditions for the sixth order equations (2–4) plus two unknowns, the displacement at a joint a, and the vertical force at a joint H.

Small deformation solutions

For small B (relatively rigid rods), useful analytic solutions are possible. We perturb about the case when each element is straight. Let

$$\phi(s) = \phi_0(s) + B\phi_1(s) + \mathcal{O}(B^2), \tag{10}$$

where ϕ may be any of the dependent variables θ, X, Y, x, y or the unknowns H and a. Without going into the details, the solution is:

$$\theta_0 = \frac{\pi}{2} + \frac{\pi}{N}, \tag{11}$$
$$Y_0 = X_0 = 0, \tag{12}$$
$$x_0 = s\cos\left(\frac{\pi}{2} + \frac{\pi}{N}\right) + \frac{1}{2}\csc\left(\frac{\pi}{N}\right), \tag{13}$$
$$y_0 = s\sin\left(\frac{\pi}{2} + \frac{\pi}{N}\right), \tag{14}$$
$$H_0 = 0, \qquad a_0 = \frac{1}{2}\csc\left(\frac{\pi}{N}\right), \tag{15}$$

$$\theta_1 = \cot\left(\frac{\pi}{N}\right)\left(c_0 + c_1 s + \frac{1}{8}s^2 - \frac{1}{12}s^3\right), \tag{16}$$
$$X_1 = \frac{1}{2}\csc\left(\frac{\pi}{N}\right)s - \frac{1}{2}\sin\left(\frac{\pi}{N}\right)s^3, \tag{17}$$
$$Y_1 = -\frac{1}{4}\cot\left(\frac{\pi}{N}\right)\csc\left(\frac{\pi}{N}\right) + \frac{1}{2}\cos\left(\frac{\pi}{N}\right)s^2, \tag{18}$$
$$x_1 = -\cot\left(\frac{\pi}{N}\right)\cos\left(\frac{\pi}{N}\right)\left(c_0 + \frac{c_1}{2}s^2 + \frac{1}{24}s^3 - \frac{1}{48}s^4\right), \tag{19}$$
$$y_1 = -\cos\left(\frac{\pi}{N}\right)\left(c_0 + \frac{c_1}{2}s^2 + \frac{1}{24}s^3 - \frac{1}{48}s^4\right), \tag{20}$$
$$H_1 = -\frac{1}{4}\cot\left(\frac{\pi}{N}\right)\csc\left(\frac{\pi}{N}\right), \qquad a_1 = 0, \tag{21}$$

where for the rigidly joined case

$$c_0 = 0, \qquad c_1 = -\frac{1}{24} \tag{22}$$

and for the hinged case

$$c_0 = -\frac{1}{48}, \qquad c_1 = 0. \tag{23}$$

The following values measure the extent of the deformation. The force occurring at the joints is

$$H = \frac{1}{4}\cot\left(\frac{\pi}{N}\right)\csc\left(\frac{\pi}{N}\right)B + \mathcal{O}(B^2). \tag{24}$$

The distance from the center of mass to a joint is

$$a = \frac{1}{2}\csc\left(\frac{\pi}{N}\right) + \mathcal{O}(B^2). \tag{25}$$

The distance from the center of mass to the midpoint of an element is

$$r = \sqrt{(x(1/2))^2 + (y(1/2))^2} = \frac{1}{2}\cot\left(\frac{\pi}{N}\right) - B\cot\left(\frac{\pi}{N}\right)\left(\frac{c_0}{2} + \frac{c_1}{8} + \frac{1}{256}\right) + \mathcal{O}(B^2). \tag{26}$$

The normalized moments at the joints and at the midpoint are

$$\theta'(0) = \cot\left(\frac{\pi}{N}\right)c_1 B + \mathcal{O}(B^2), \tag{27}$$

$$\theta'(1/2) = \cot\left(\frac{\pi}{N}\right)\left(c_1 + \frac{1}{16}\right)B + \mathcal{O}(B^2). \tag{28}$$

Notice that for the hinged case the maximum magnitude of the bending moment occurs at the midpoint, while for the rigidly joined case the maximum occurs at the joints.

Numerical integration

When B is not small numerical integration is necessary. This can be achieved by a variety of methods including finite difference, collocation, or finite element schemes. For robustness and ease of computation, we elect to use a combination of quasi-Newton and homotopy[2] methods similar to those used in the rotating ring problem[1]. Define the vector of unknowns

$$v = \left(a, \quad -h, \quad \theta(0) \text{ or } \frac{d\theta}{ds}(0)\right)^t, \tag{29}$$

and let $x(s;v)$, $y(s;v)$, $X(s;v)$, $Y(s;v)$, $\theta(s;v)$ be the solution of the initial value problem given by Eqs. (2–4) with initial conditions given by Eqs. (5), (6), (8) (or (9)), and (29). The original two-point boundary value problem is equivalent to solving the nonlinear system of equations

$$F(v) = \begin{pmatrix} y\left(\frac{1}{2};v\right) - x\left(\frac{1}{2};v\right)\tan\left(\frac{\pi}{N}\right) \\ \theta\left(\frac{1}{2};v\right) - \left(\frac{\pi}{2} + \frac{\pi}{N}\right) \\ \frac{d^2\theta}{ds^2}\left(\frac{1}{2};v\right) \end{pmatrix} = 0. \tag{30}$$

Algorithms for solving the nonlinear system (30) typically require partial derivatives $\partial F_i/\partial v_j$. These partials can be easily estimated by solving a system of ordinary differential equations similar to those described earlier in Watson and Wang[1]. The strategy then is to use the globally convergent and robust homotopy method[2] to get a few solutions, and then use the inexpensive (but only locally convergent) quasi-Newton method to generate many solutions as B is slowly varied.

Results and discussion

For large N and/or large B, the frame approaches a circular shape. The asymptotic limits are

$$r \to \frac{N}{2\pi}, \quad a \to \frac{N}{2\pi}, \quad \theta' \to \frac{2\pi}{N}, \quad H \to \frac{BN^2}{4\pi^2}. \tag{31}$$

Figure 2 shows the maximum normalized bending moment, occurring at the midpoint, for the hinged case. Our approximate formulas for small and large B compare well with numerical results in their respective regions of validity. Figure 3 shows the maximum moment of a rigidly joined frame, now occurring at the joint instead of at the midpoint. This indicates failure due to bending is quite sensitive to the joining method.

Figure 4 shows the deformations of a rigidly joined triangular frame under free rotation. The hexagonal frame is shown in Fig. 5. Our results could also be applied to hollow polygonal cylinders.

References

[1] Watson, L. T. and Wang, C.-Y., "Free rotation of a circular ring with an unbalanced mass," *AIAA Journal*, Vol. 27, Nov. 1989, pp. 1650–1652.

[2] Watson, L. T., Billups, S. C., and Morgan, A. P., "HOMPACK: A suite of codes for globally convergent homotopy algorithms," *ACM Trans. Math. Software*, Vol. 13, 1987, pp. 281–310.

Figure Captions

Figure 1. a) The rotating polygonal frame. b) The coordinate system. c) An elemental length.

Figure 2. Maximum normalized moment, hinged case. Dashed lines are approximations from Eq. (28) or (31).

Figure 3. Maximum normalized moment, rigidly joined case. Dashed lines are approximations from Eq. (27).

Figure 4. Deformations for $N = 3$, rigidly joined case.

Figure 5. Deformations for $N = 6$, rigidly joined case.

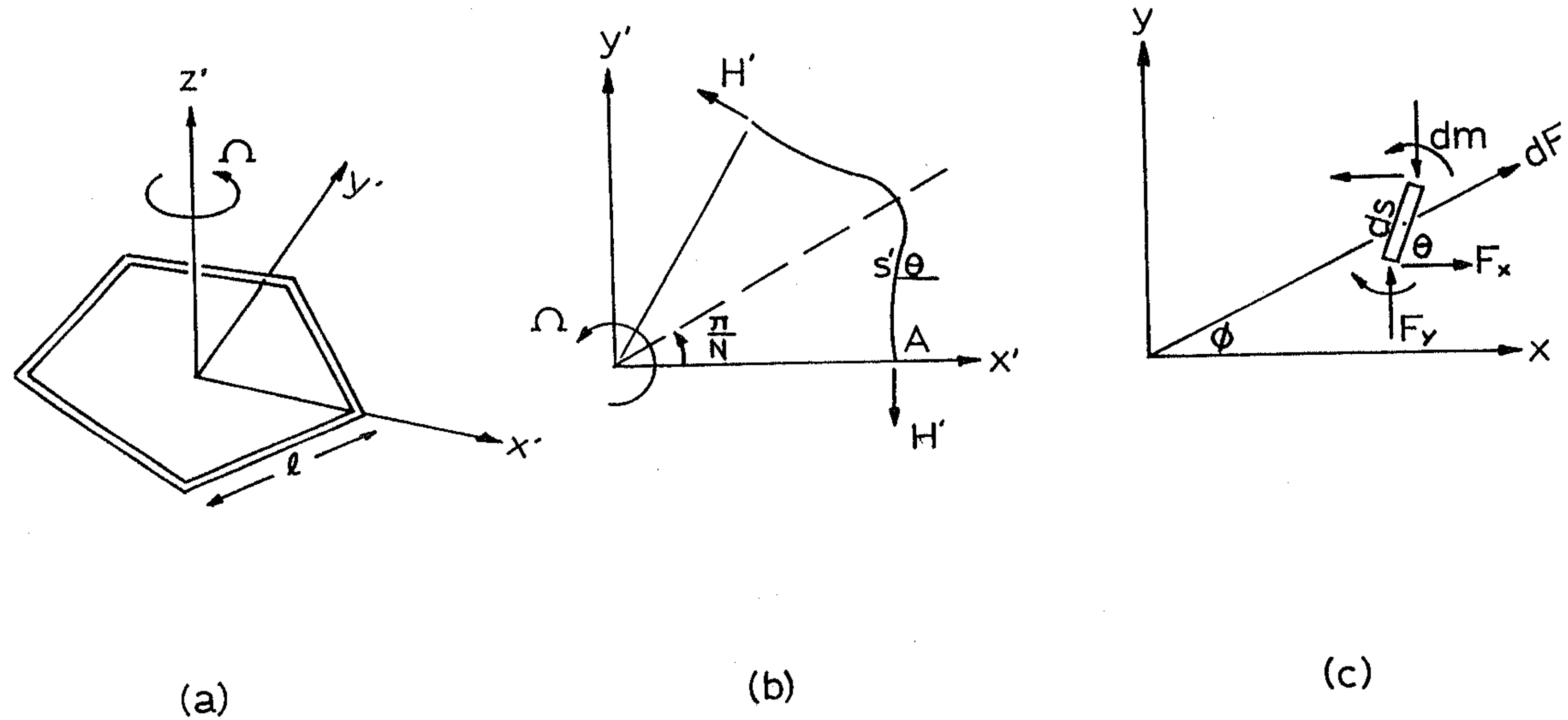

Fig 1

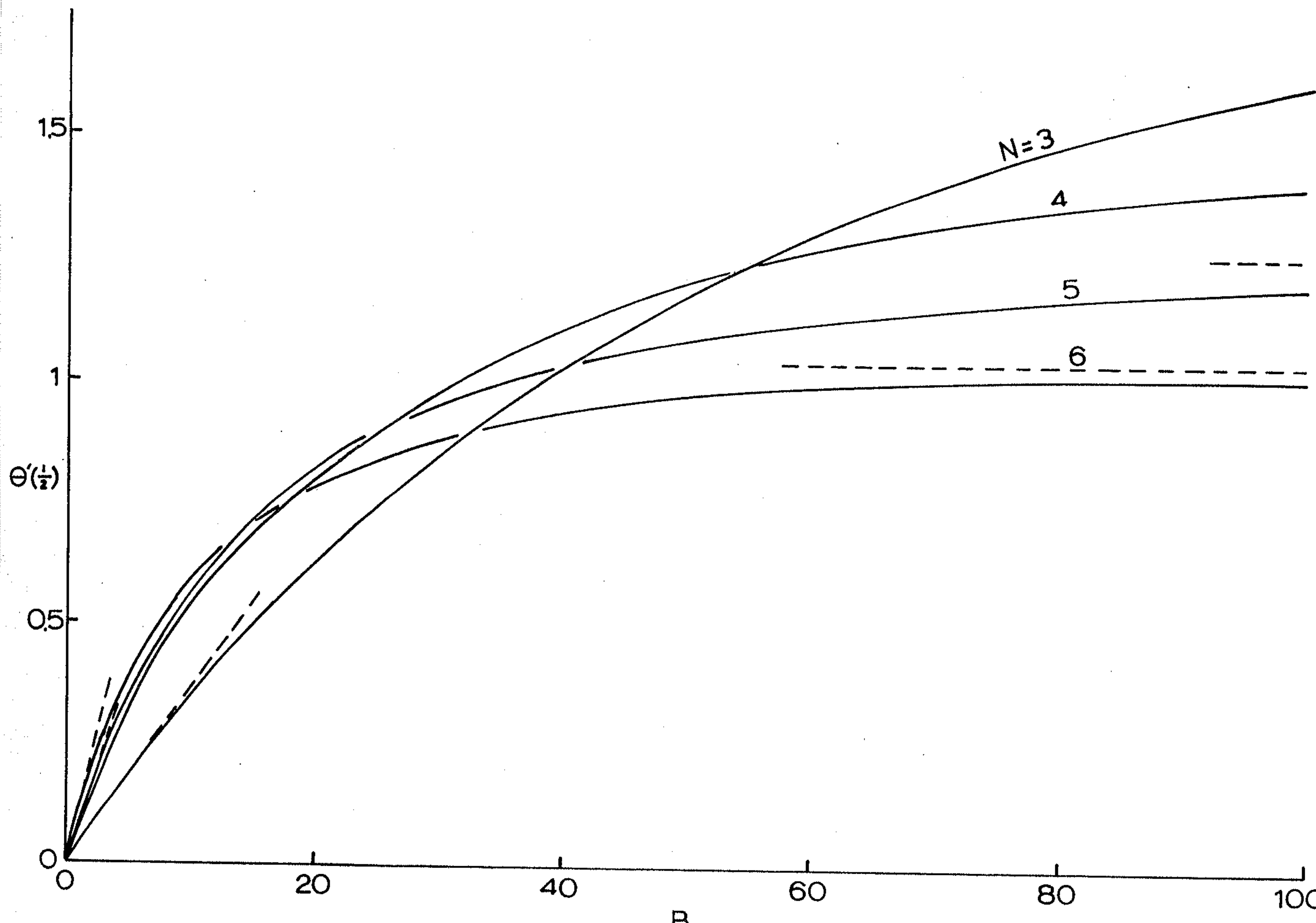

Fig 2

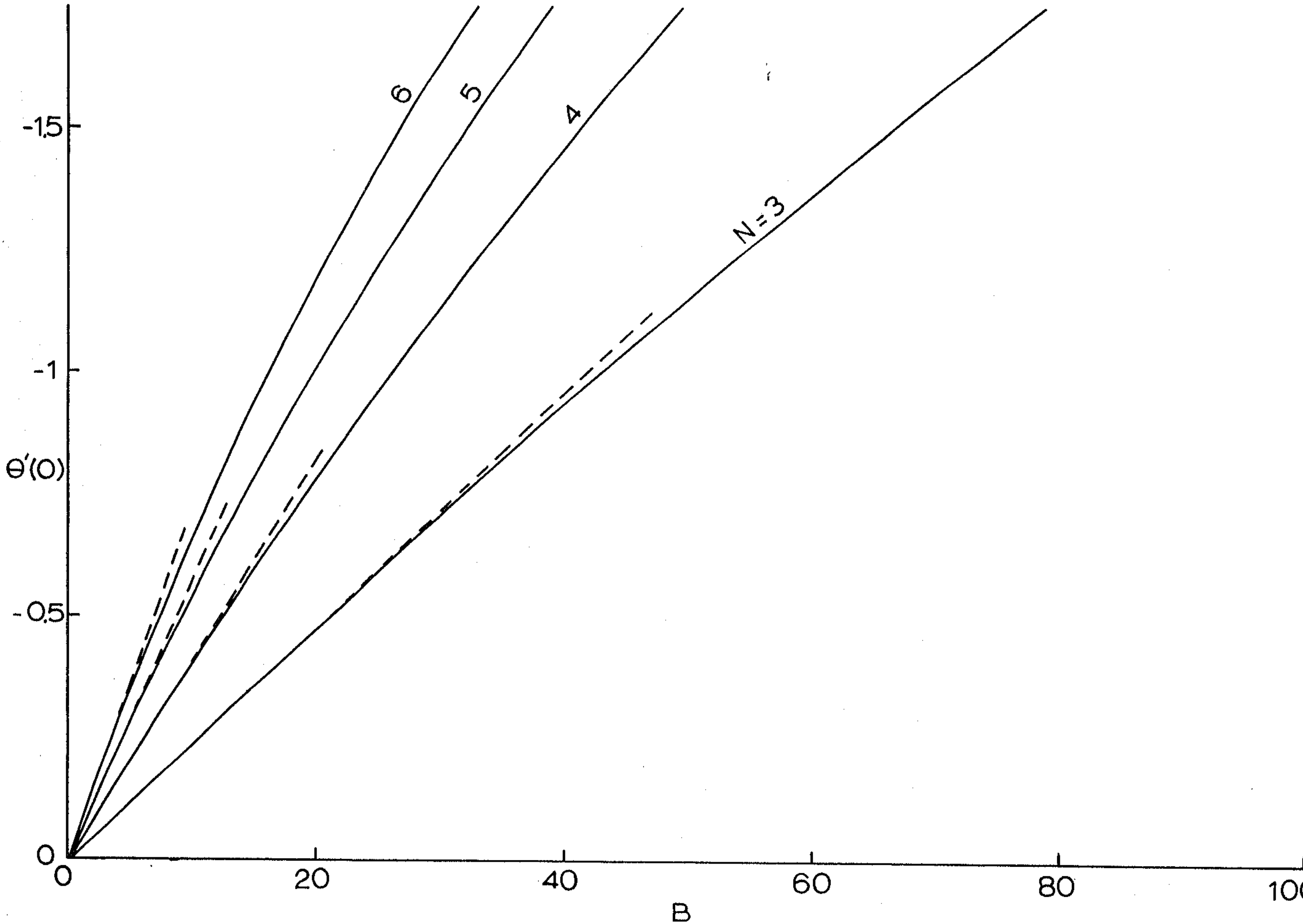

Fig 3

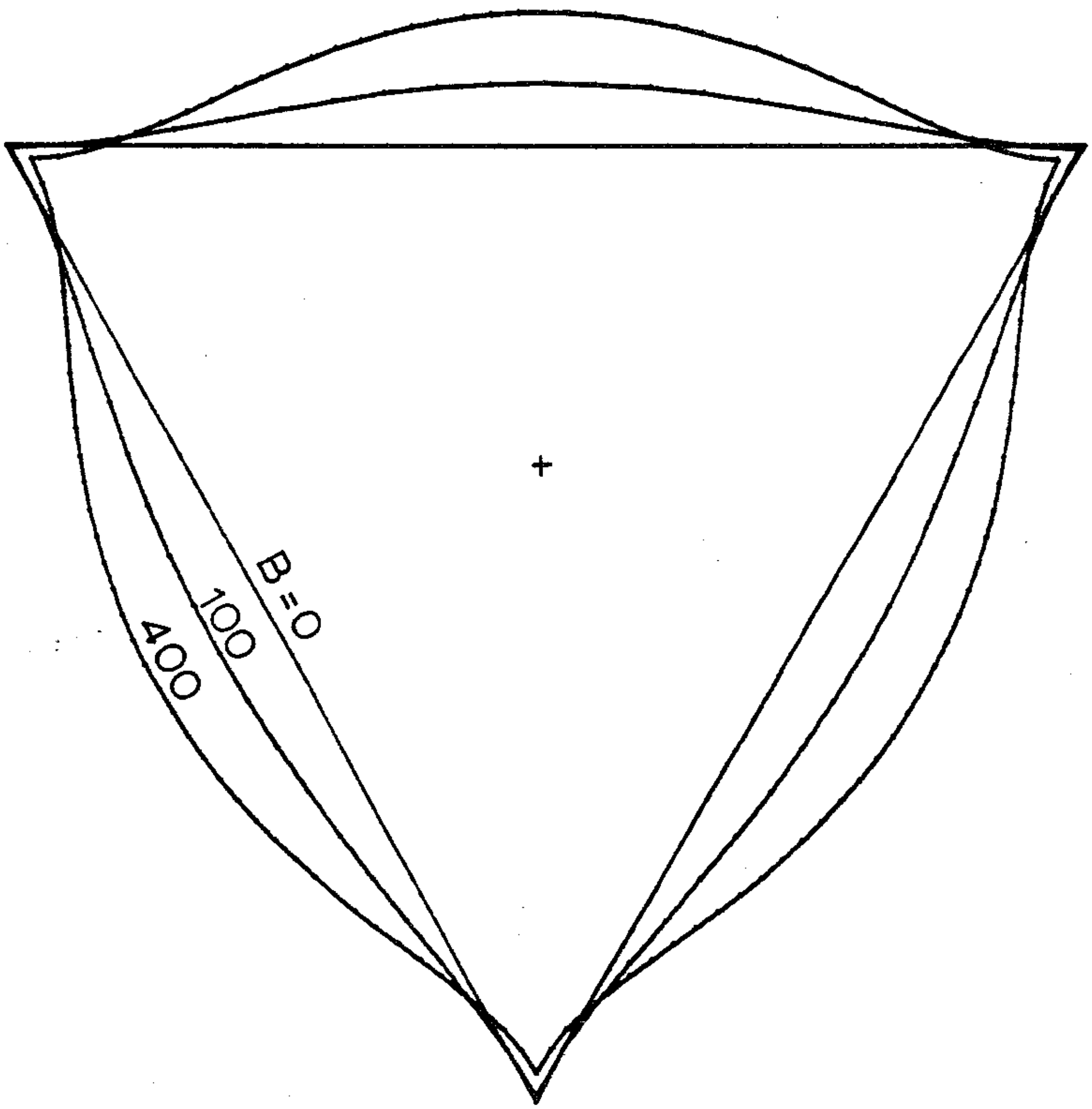

Fig 4

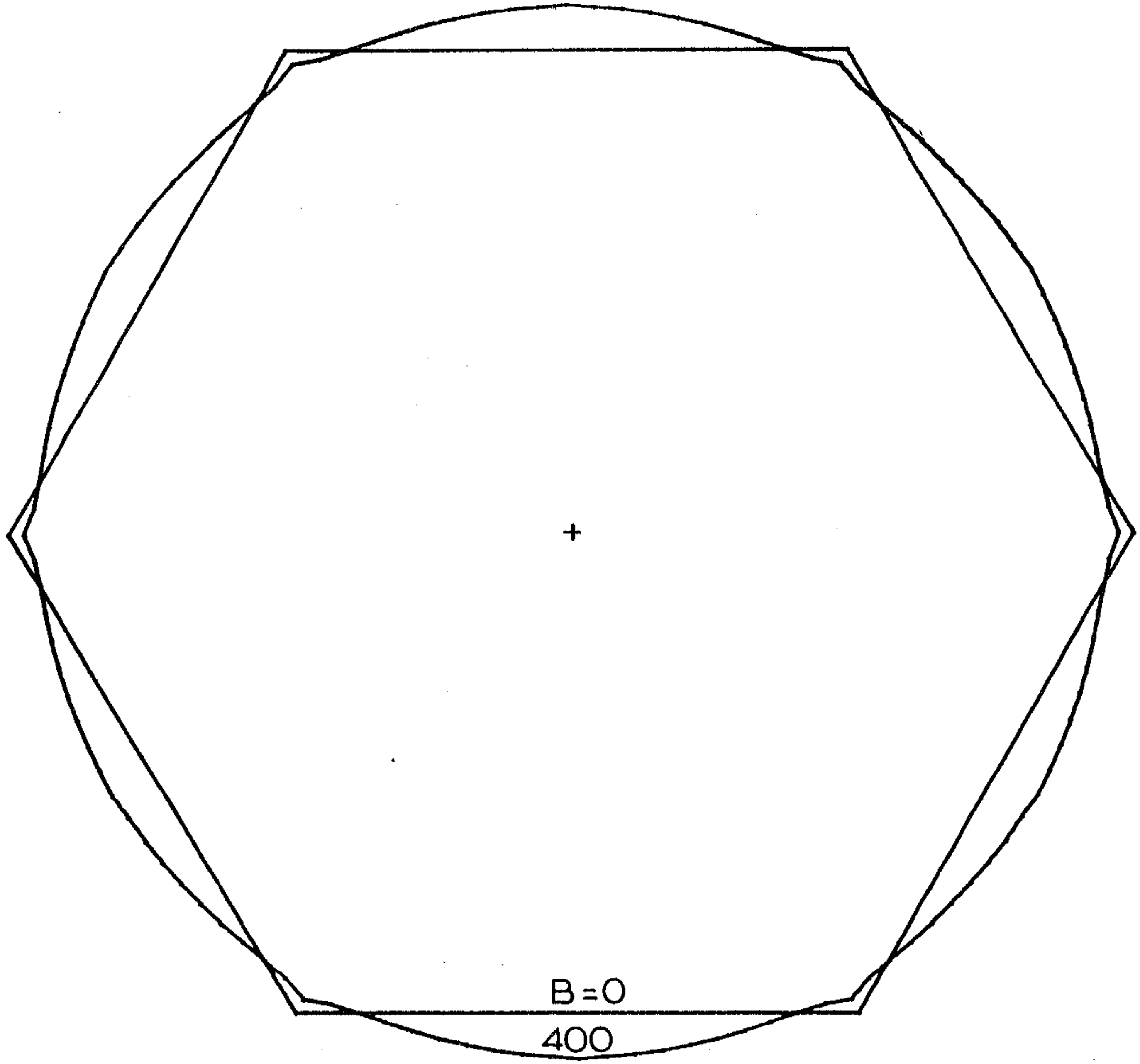

Fig.5