An Active Set Algorithm for Tracing Parametrized Optima

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AN ACTIVE SET ALGORITHM FOR TRACING PARAMETRIZED OPTIMA

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Abstract. Optimization problems often depend on parameters that define constraints or objective functions. It is often necessary to know the effect of a change in a parameter on the optimum solution. An algorithm is presented here for tracking paths of optimal solutions of inequality constrained nonlinear programming problems as a function of a parameter. The proposed algorithm employs homotopy zero-curve tracing techniques to track segments where the set of active constraints is unchanged. The transition between segments is handled by considering all possible sets of active constraints and eliminating nonoptimal ones based on the signs of the Lagrange multipliers and the derivatives of the optimal solutions with respect to the parameter. A spring-mass problem is used to illustrate all possible kinds of transition events, and the algorithm is applied to a well known ten-bar truss structural optimization problem.

1. Introduction.

Optimization problems often depend on parameters that define constraint boundaries or objective function properties. These parameters are kept constant during the optimization, but it is often necessary to know what is the effect of a change in some parameter on the optimum solution. An example is the design of a structure subject to stress constraints, where the stress limit may be a parameter that can be varied by using lower-grade or better-grade metal alloys. Another example occurs in optimization with two objective functions, where all the efficient solutions can be obtained by minimizing all the convex combinations of the two objective functions. The parameter of interest there controls the relative proportions of the two objective functions in the combination. The dependence of the optimum on the problem parameter can also be helpful in the modeling process, since singularities in the behavior of the system can be revealed in this way [1].

There has been substantial interest in calculating the derivatives of optima with respect to such parameters (e.g., [2]). More recently there has been an effort to develop an approach to tracing the family of optima obtained by varying a parameter over an extended range by using homotopy techniques ([4], [3]). Reference [4] demonstrated that it is possible to trace the optimal path when the optimum solution for the initial value of the parameter is given. The optimum path is composed of smooth segments which are connected at transition points where the set of active constraints changes. The main challenge is to develop techniques for making the transition from one segment to the next. In [4] such a technique was developed for the design of a beam with a given weight so as to maximize the buckling load with the weight being the varied parameter. The objective of this paper is to develop a general algorithm for tracing the optima of inequality constrained optimization problems as a function of a parameter.

2. Problem statement.

We want to minimize a cost function $c(\mathbf{k}, \theta)$ subject to constraints $G_j(\mathbf{k}, \theta) \leq 0$, $j = 1, \ldots, n_2$, where $\mathbf{k}$ is a n_1-vector of design variables subject to the minimum value constraints $k_i \geq k_{0i}$ and θ is a parameter. The problem is formulated as

$$\text{minimize } c(\mathbf{k}, \theta) \tag{1}$$

subject to

$$G_i = k_{0i} - k_i \leqq 0, \qquad i = 1, \ldots, n_1, \tag{2}$$

$$G_{j+n_1}(\mathbf{k}, \theta) \leqq 0, \qquad j = 1, \ldots, n_2. \tag{3}$$

The solution should be obtained for a specified range of θ, say $\theta_a \leqq \theta \leqq \theta_b$. The Lagrangian function and Kuhn-Tucker conditions for this problem are:

$$L(k, \theta, \lambda) = c(\mathbf{k}) + \sum_{i=1}^{n_1} \lambda_i (k_{0i} - k_i) + \sum_{j=n_1+1}^{n_1+n_2} \lambda_j G_j(\mathbf{k}, \theta) \tag{4}$$

$$\frac{\partial c}{\partial k_i} + \sum_{j=n_1+1}^{n_1+n_2} \lambda_j \frac{\partial G_j}{\partial k_i} - \lambda_i = 0, \qquad i = 1, \ldots, n_1, \tag{5}$$

$$G_j \lambda_j = 0, \qquad j = 1, \ldots, n_1 + n_2, \tag{6}$$

$$\lambda_j \geqq 0, \qquad j = 1, \ldots, n_1 + n_2, \tag{7}$$

$$G_j \leqq 0, \qquad j = 1, \ldots, n_1 + n_2. \tag{8}$$

Equations (5)–(6) form a system of nonlinear equations to be solved for the design variables k_i and for the Lagrange multipliers λ_j associated with active constraints of the form (3) and with the bounds for design variables (2). The solution of these equations is a function of θ. As the value of θ increases the solution of the Kuhn-Tucker conditions follows a path that consists of several smooth segments, each segment characterized by a different set of active constraints.

3. Homotopy method.

The system of nonlinear equations (5)–(6) is solved by a homotopy method. The homotopy method uses the fact that if the solution to the system of equations

$$\mathbf{F}(\mathbf{x}, \theta) = 0 \tag{9}$$

is known at some point $(\mathbf{x}_0, \theta_0)$, and the Jacobian matrix $\mathbf{DF}(\mathbf{x}_0, \theta_0)$ of the function $\mathbf{F}$ at $(\mathbf{x}_0, \theta_0)$ has full rank, then there is some neighbourhood U of $(\mathbf{x}_0, \theta_0)$ such that there is a unique curve of zeros of $\mathbf{F}(\mathbf{x}, \theta)$ in U passing through $(\mathbf{x}_0, \theta_0)$. According to the theory in [6], [7], this full rank of the Jacobian matrix implies that the zero set of equations (9) contains a smooth curve Γ in $(N+1)$-dimensional $(\mathbf{x}, \theta)$ space, which has no bifurcations and is disjoint from other zeros of (9). The curve Γ can be parametrized by the arc length s as

$$\mathbf{x} = \mathbf{x}(s), \qquad \theta = \theta(s). \tag{10}$$

Taking the derivative of (9) with respect to arc length, the nonlinear system of equations is transformed into a set of ordinary differential equations

$$\left[\mathbf{F}_x(\mathbf{x}(s),\theta(s)),\quad \mathbf{F}_\theta(\mathbf{x}(s),\theta(s))\right]\begin{pmatrix}\frac{d\mathbf{x}}{ds}\\ \frac{d\theta}{ds}\end{pmatrix}=0, \tag{11}$$

and

$$\left\|\begin{pmatrix}\frac{d\mathbf{x}}{ds}\\ \frac{d\theta}{ds}\end{pmatrix}\right\|=1, \tag{12}$$

where $\mathbf{F}_x$ and $\mathbf{F}_\theta$ denote the partial derivatives of $\mathbf{F}$ with respect to $\mathbf{x}$ and θ respectively. With the initial conditions at $s=0$,

$$\mathbf{x}(0)=\mathbf{x}_0,\qquad \theta(0)=\theta_a, \tag{13}$$

(11)–(13) can be treated as an initial value problem. Its trajectory is the path Γ of optimal solutions $Z(s)=(\mathbf{x}(s),\theta(s))$.

A software package HOMPACK ([7]) which implements several homotopy algorithms can be used to track the zero curve Γ. The HOMPACK algorithms take steps along the zero curve using prediction and correction to find the next point. In the prediction phase a Hermite cubic $p(s)$ is constructed which interpolates the zero curve Γ at two known points $Z(s_1)$ and $Z(s_2)$.

The predicted next point is

$$Z^{(0)}=p(s_2+h), \tag{14}$$

where $p(s)$ is the Hermite cubic, and h is an estimate of the optimal step (in arc length) to take along Γ.

The corrector iteration is

$$Z^{(k+1)}=Z^{(k)}-[\mathbf{DF}(Z^{(k)})]^{+}\mathbf{F}(Z^{(k)}),\qquad k=0,1,\ldots$$

where $[\mathbf{DF}(Z^{(k)})]^{+}$ is the Moore-Penrose pseudoinverse of the $n\times(n+1)$ Jacobian matrix $\mathbf{DF}$. In practice this pseudoinverse is not calculated explicitly; see [7] for the details of the Hermite cubic interpolant construction and the corrector iteration.

The optimal step size h is chosen to prevent the correction iteration from being too costly. The user can specify nondefault values used in determining the step size such as, for example, the maximum and minimum allowed step size. The parameter θ in equations (11)–(13) is a dependent variable, which distinguishes homotopy methods from standard continuation, imbedding, or incremental methods. The homotopy approach is also different from initial value or differentiation methods, since the controlling variable is arc length s, rather than θ.

4. Solution along a segment.

Since the active constraints in a segment are fixed they can be treated as equality constraints. Furthermore along each segment some design variables are fixed at their lower bound. The vector of these inactive (passive) variables is denoted $\mathbf{k_p}$ and need not be considered as design variables for that segment. The vector of active design variables k_i ($i \in \mathcal{I}_a$) is denoted as $\mathbf{k_a}$. Along each segment the Kuhn-Tucker conditions are solved for the active design variables ($k_i \in \mathbf{k_a}$) and for the Lagrange multipliers associated with the active constraints of the form (3) ($\lambda_j \in \lambda_\mathbf{g}$, $j \in \mathcal{I}_g$). For each segment there are 2 types of equations:

$$V1 : G_j(\mathbf{k}, \theta) = 0, \qquad j \in \mathcal{I}_g, \tag{15}$$

$$V2 : \frac{\partial c}{\partial k_i} + \sum_{j \in \mathcal{I}_g} \lambda_j \frac{\partial G_j}{\partial k_i} = 0, \qquad i \in \mathcal{I}_a. \tag{16}$$

The active design variables and the Lagrange multipliers associated with active constraints (3) are the variables in these equations. For the homotopy solution we need the Jacobian matrix of these functions with respect to θ, $\mathbf{k_a}$, and $\lambda_\mathbf{g}$. The Jacobian matrix has components of the following form:

$$\frac{\partial V1}{\partial \theta} = \frac{\partial G_j}{\partial \theta}, \qquad \frac{\partial V1}{\partial k_m} = \frac{\partial G_j}{\partial k_m}, \qquad \frac{\partial V1}{\partial \lambda_t} = 0,$$

$$\frac{\partial V2}{\partial \theta} = \frac{\partial^2 c}{\partial \theta \partial k_i} + \sum_{j \in \mathcal{I}_g} \lambda_j \frac{\partial^2 G_j}{\partial \theta \partial k_i}, \quad \frac{\partial V2}{\partial k_m} = \frac{\partial^2 c}{\partial k_i \partial k_m} + \sum_{j \in \mathcal{I}_g} \lambda_j \frac{\partial^2 G_j}{\partial k_i \partial k_m}, \quad \frac{\partial V2}{\partial \lambda_j} = \frac{\partial G_j}{\partial k_i}, \tag{17}$$

where $i, m \in \mathcal{I}_a$ and $t, j \in \mathcal{I}_g$. The derivatives with respect to k_i and k_m denote the derivatives with respect to all active design variables, and the derivatives with respect to λ_t and λ_j denote the derivatives with respect to all Lagrange multipliers associated with active constraints of the form (3).

For example, for $n_1 = 5$ with active constraint G_{n_1+2} of the form (3) and active design variables k_1, k_5 the system of equations is:

$$\frac{\partial c}{\partial k_1} + \lambda_7 \frac{\partial G_7}{\partial k_1} = 0, \tag{18}$$

$$\frac{\partial c}{\partial k_5} + \lambda_7 \frac{\partial G_7}{\partial k_5} = 0, \tag{19}$$

$$G_7(k_1, k_5, \theta) = 0. \tag{20}$$

The set of unknown variables is ordered as $(\theta, k_1, k_5, \lambda_7)$ and the corresponding Jacobian matrix is:

$$\begin{pmatrix} \frac{\partial^2 c}{\partial \theta \partial k_1} + \lambda_7 \frac{\partial^2 G_7}{\partial \theta \partial k_1} & \frac{\partial^2 c}{\partial k_1^2} + \lambda_7 \frac{\partial^2 G_7}{\partial^2 k_1} & \frac{\partial^2 c}{\partial k_1 \partial k_5} + \lambda_7 \frac{\partial^2 G_7}{\partial k_1 \partial k_5} & \frac{\partial G_7}{\partial k_1} \\ \frac{\partial^2 c}{\partial \theta \partial k_5} + \lambda_7 \frac{\partial^2 G_7}{\partial \theta \partial k_5} & \frac{\partial^2 c}{\partial k_1 \partial k_5} + \lambda_7 \frac{\partial^2 G_7}{\partial k_1 \partial k_5} & \frac{\partial^2 c}{\partial^2 k_5} + \lambda_7 \frac{\partial^2 G_7}{\partial^2 k_5} & \frac{\partial G_7}{\partial k_5} \\ \frac{\partial G_7}{\partial \theta} & \frac{\partial G_7}{\partial k_1} & \frac{\partial G_7}{\partial k_5} & 0 \end{pmatrix}. \tag{21}$$

The variables of the system of equations are stored in the vector of homotopy variables $\mathbf{y} = \begin{pmatrix} \theta \\ \mathbf{k_a} \\ \lambda_\mathbf{g} \end{pmatrix}$.

At the start of a segment the set of active design variables and active constraints for this segment has to be found, so that the vector $\mathbf{y}$ be defined. A set of equations is then generated, with the type of each variable determining the form of the equation appended to the system of equations. For the Lagrange multiplier associated with an active constraint of the form (3) the equation has the form (15), and for an active design variable the equation has the form (16). The Jacobian matrix is calculated according to (17) and the system of equations for the segment is solved using the homotopy technique. Then the Lagrange multipliers for inactive design variables are calculated according to (5). In these equations the Lagrange multipliers associated with active constraints of the form (3) have been computed by the homotopy method, and the Lagrange multipliers associated with inactive constraints (3) are known to be zero.

5. Segment termination and transition to the next segment.

At each point of a segment the Lagrange multipliers associated with the lower bound of the inactive design variables or the active constraints of the form (3) in the segment should be nonnegative, the value of each G_j, $j = n_1, \ldots, n_1 + n_2$ should be less than or equal to zero, and all design variables should be larger than or equal to their lower bound. If any of the above conditions is not satisfied the segment is terminated and a new one is started. The transition point to a new segment is called here a *switching point.* (It is assumed throughout that a switching point is not also a turning point of the path of optima.) Depending on the type of termination, the switching point is the point where

1) one of the positive Lagrange multipliers becomes equal to zero, or
2) a previously negative G_j of the form (3) becomes equal to zero, or
3) an active design variable ($k_i \in \mathbf{k_a}$) becomes inactive (equal to k_{0i}).

At the beginning of each segment the system of linear equations (5) is solved for $\lambda_1, \ldots, \lambda_m$, $m = n_1 + n_2$, to check which design variables and constraints are active and to find the initial values of the Lagrange multipliers for the segment. First the Lagrange multipliers for inactive constraints are set to zero so that we consider Lagrange multipliers only for potentially active constraints (those equal to zero).

Since some of the constraints (3) may be inactive (their values at the switching point are less than zero), or the derivatives of the constraints (3) with respect to the design variables can assume values for which some columns or rows in the coefficient matrix of the system (5) are linearly dependent, the rank of this matrix can be less than n_2. The rank of the coefficient matrix for the system (5) determines the number of the constraints (3) which are assumed to be active in the next segment.

The QR factorization with column pivoting is used to find the rank (r) of the coefficient matrix. Next the system (5) is solved for all subsets of r columns which are linearly independent assuming that the Lagrange multipliers for the constraints (3) corresponding to the remaining columns are zero. To get the solution for each subset at least r design variables are assumed to be active (the corresponding Lagrange multipliers are set to zero). For each subset of r columns (corresponding to r constraints) all combinations of r out of n_1 design variables are assumed to be active. The system is solved in turn for each combination to find all sets of active design variables and active constraints (3) such that the Lagrange multipliers are nonnegative.

Sometimes there are several solutions satisfying the condition that all the Lagrange multipliers be nonnegative. Then for each solution the derivatives of the design variables with respect to the arc length s are calculated. A set of active constraints (3) and active design variables is accepted when the values of these derivatives indicate that no active constraint will be violated for increasing values of s.

To calculate the values of the derivatives of the design variables with respect to θ the Kuhn–Tucker conditions (5)–(6) are differentiated with respect to θ. Thus we obtain:

$$(\mathbf{A}+\mathbf{Z})\frac{\partial \mathbf{k_a}}{\partial \theta} + \mathbf{N}\frac{\partial \lambda_\mathbf{g}}{\partial \theta} + \frac{\partial(\nabla \mathbf{c})}{\partial \theta} + (\frac{\partial \mathbf{N}}{\partial \theta})\lambda_\mathbf{g} = \mathbf{0}, \tag{22}$$

$$\mathbf{N}^\mathrm{T}\frac{\partial \mathbf{k_a}}{\partial \theta} + \frac{\partial \mathbf{G_a}}{\partial \theta} = \mathbf{0}, \tag{23}$$

where $\mathbf{k_a}, \lambda_\mathbf{g}$ are a vector of design variables and a vector of the Lagrange multipliers for active G_j, $\mathbf{G_a}$ is a vector of active constraints G_j, $j \in \mathcal{I}_g$, $\mathbf{N}$ has components $n_{ij} = \dfrac{\partial G_j}{\partial k_i}$, $(j \in \mathcal{I}_g,\ i \in \mathcal{I}_a)$, $\mathbf{A}$ is the Hessian of the objective function c, $a_{ij} = \dfrac{\partial^2 c}{\partial k_i \partial k_j}$, and $\mathbf{Z}$ is a matrix with elements $z_{il} = \sum_{j\in\mathcal{I}_g} \dfrac{\partial^2 G_j}{\partial k_i \partial k_l}\lambda_j$. After equations (22) and (23) are solved, derivatives of each G_j corresponding to an active constraint (3) with respect to θ are calculated according to

$$\frac{\partial G_j}{\partial \theta} = \sum_{i\in\mathcal{I}_a} \frac{\partial G_j}{\partial k_i}\frac{\partial k_i}{\partial \theta}, \qquad j \in \mathcal{I}_g. \tag{24}$$

For each candidate solution satisfying the Kuhn-Tucker conditions, derivatives with respect to arc length s are then calculated by multiplication by $d\theta/ds$ (taken from the previously calculated point on the segment–all that matters is the sign of $d\theta/ds$). We calculate the derivatives with respect to arc length s of design variables, Lagrange multipliers and G_j's corresponding to active constraints. A solution is accepted if the derivatives with respect to s of active design variables that are at their lower bound are nonnegative, the derivatives with respect to s of zero Lagrange multipliers that correspond to active constraints (3) are nonnegative and the derivatives of G_j's that are equal to zero are nonpositive.

The path of optimal points can be discontinuous [1], [3]. It is possible that beyond some value of θ there are no neighbouring optima. At this point θ is fixed and the problem must be solved by a standard optimization algorithm to find a new optimum. A path of optimal solutions can then be resumed at this new point and followed as before. It is also possible that beyond a certain value of θ no optimum exists, for example, if the problem becomes unbounded. A more detailed description of this segment switching algorithm is given in the Appendix.

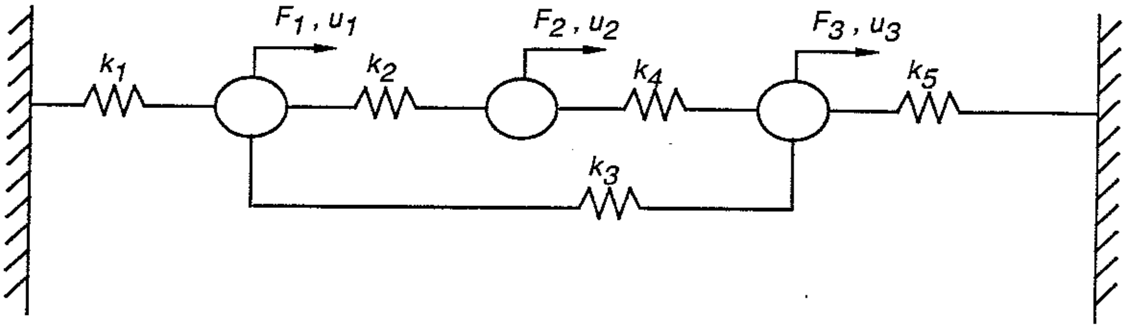

Figure 1. Spring-mass system.

6. Spring-mass system example.

6.1. Problem definition.

Consider the mass-spring system in Figure 1 including 5 springs and 3 masses. Let $c_1, \ldots, c_5 > 0$ be costs associated with the springs, u_1, u_2, u_3 be the mass displacements and $k_1, \ldots, k_5$ the spring constants. The objective is to vary the spring constants so as to minimize the cost of the springs subject to the condition that displacements are bounded in magnitude by u_a. We want to find the dependence of the optimum solution on the displacement limit u_a. This simple problem permits us to generate a variety of segment transition scenarios by varying the spring constants and applied forces. The problem is formulated as

$$\text{minimize } c(\mathbf{k}) = c_1 k_1 + c_2 k_2 + c_3 k_3 + c_4 k_4 + c_5 k_5 \tag{25}$$

subject to

$$G_i = 1 - k_i \leqq 0, \quad i = 1, \ldots, 5, \tag{26}$$

$$G_6 = -u_a + u_1(\mathbf{k}, \mathbf{F}) \leqq 0, \quad G_7 = -u_a + u_2(\mathbf{k}, \mathbf{F}) \leqq 0, \quad G_8 = -u_a + u_3(\mathbf{k}, \mathbf{F}) \leqq 0, \tag{27}$$

$$G_9 = -u_a - u_1(\mathbf{k}, \mathbf{F}) \leqq 0, \quad G_{10} = -u_a - u_2(\mathbf{k}, \mathbf{F}) \leqq 0, \quad G_{11} = -u_a - u_3(\mathbf{k}, \mathbf{F}) \leqq 0, \tag{28}$$

where $\mathbf{F}$ is the force vector and $\mathbf{k}$ is the vector of spring constants.

The displacements u_i are obtained by solving the equilibrium equations

$$\mathbf{K}\mathbf{u} = \mathbf{F}, \tag{29}$$

where $\mathbf{K}$ is the stiffness matrix related to the spring constants:

$$\mathbf{K} = \begin{pmatrix} k_2 + k_1 + k_3 & -k_2 & -k_3 \\ -k_2 & k_2 + k_4 & -k_4 \\ -k_3 & -k_4 & k_5 + k_3 + k_4 \end{pmatrix}. \tag{30}$$

The solution needs to be obtained for all values of allowable displacement u_a. The homotopy parameter is taken as $\theta = 1/u_a$, $(0 < \theta < \infty)$. The Lagrangian function and Kuhn-Tucker conditions for this problem are:

$$L = \sum_{i=1}^{5} c_i k_i + \sum_{i=1}^{5} \lambda_i (1 - k_i) + \sum_{j=6}^{8} \lambda_j (u_{j-5} - 1/\theta) + \sum_{l=9}^{11} \lambda_l (-u_{l-8} - 1/\theta), \tag{31}$$

$$c_i + \sum_{j=6}^{8} \lambda_j \frac{\partial u_{j-5}}{\partial k_i} - \sum_{j=9}^{11} \lambda_j \frac{\partial u_{j-8}}{\partial k_i} - \lambda_i = 0, \qquad i = 1, \ldots, 5, \tag{32}$$

$$G_j \lambda_j = 0, \qquad j = 1, \ldots, 11, \tag{33}$$

$$\lambda_j \geqq 0, \qquad j = 1, \ldots, 11, \tag{34}$$

$$G_j \leqq 0, \qquad j = 1, \ldots, 11. \tag{35}$$

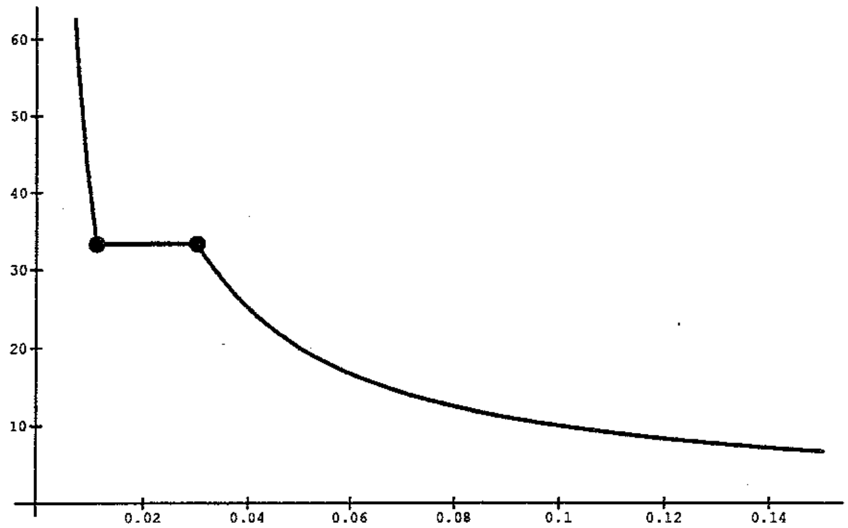

Figure 2. Displacement $u_1(\theta)$ from Table 2.

6.2. Results.

Paths of optimal solutions for the spring problem for different cost coefficients and sets of forces selected so as to bring about different switching point scenarios are described in Tables 1–7. For small values of θ (large values of the allowable displacement u_a) none of the displacement constraints are active. At the beginning of the first segment all design variables are assumed to be at their lower bound ($k_0 = 1.0$) and the greatest displacement is assumed to be active. The reciprocal of the magnitude of the greatest displacement is the starting value for the homotopy parameter θ. Next the value of the θ is increased and the optimal values of the design variables and the Lagrange multipliers associated with active displacements are computed using the homotopy method.

The path shown in Table 1 consists of five segments. Segments were terminated when a design variable became active or when a constraint for a displacement became active. The design variable k_1, which was decreasing in segments 2 and 3, starts from its lower bound value in segment 4.

The path in Table 2 contains three segments, shown in Figure 2 for the displacement u_1 (the large solid dots mark the transition points). For both switching points in this table a design variable and a constraint for a displacement became active simultaneously. The new design variable in the new segment was chosen by considering all possible sets of active design variables according to the procedure described in Section 5. Note that the initial value of the Lagrange multiplier λ_8 (for constraint on u_3) in segment 2 differs from its end value in the previous segment.

The path in Table 3 consists of four segments. The cost coefficients in Table 3 have been chosen to get a switching point where two variables (k_4 and k_5) become active at the same time (segment 1 – segment 2). Two other switching points are the points where the constraints for a displacement became active.

In Table 4 all three displacements and three spring constants become active at the starting point. The path of optimal solutions contains only one segment. The active design variables in this segment have been found by considering all possible sets of active variables.

The path in Table 5 consists of two segments. At the switching point two spring constants and two displacements become active simultaneously.

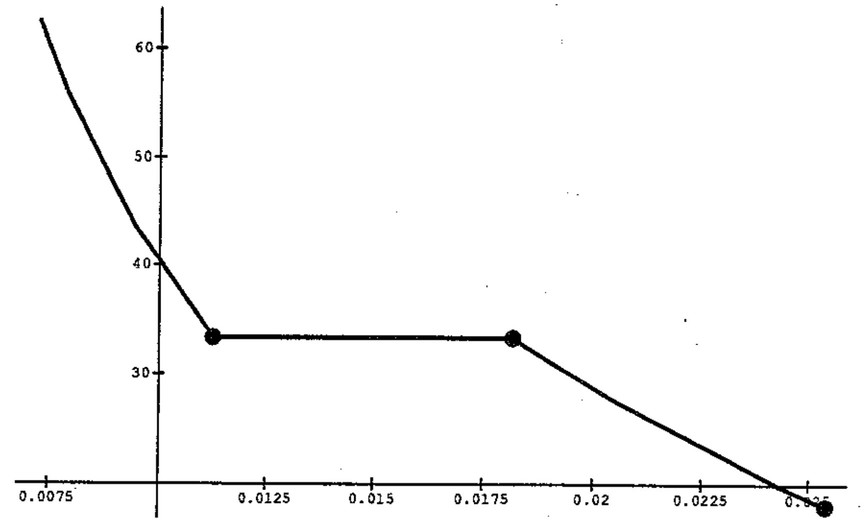

Figure 3. Displacement $u_1(\theta)$ from Table 6.

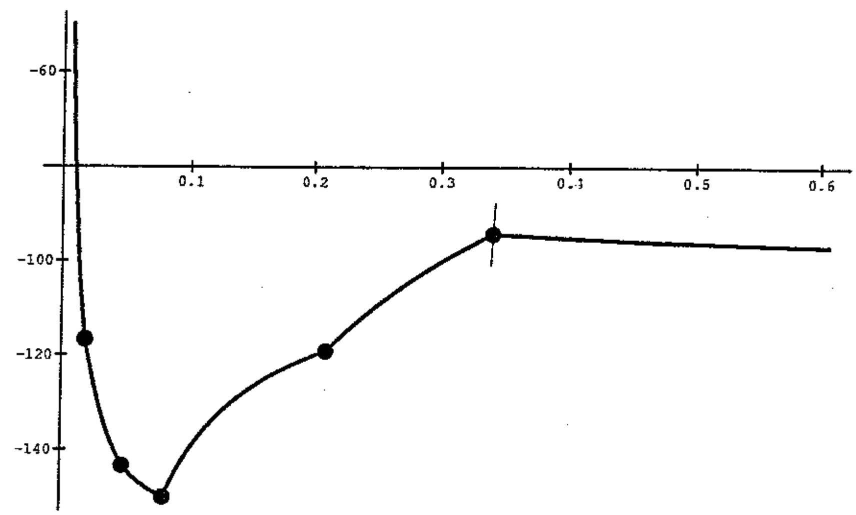

Figure 4. Displacement $u_2(\theta)$ from Table 7.

In Table 6 the cost coefficient for spring 5 is a strongly decreasing function of θ and becomes negative in segment 3. The path (see Figure 3) contains three segments. For the first switching point a design variable (k_4) and a constraint for displacement u_2 become active simultaneously. At the next switching point another spring constant (k_2) becomes active. For $\theta > 0.0254071$ the problem becomes unbounded and the cost function could be decreased indefinitely for increasing values of k_2, k_4 and k_5.

In Table 7 the constraints on u_2 were changed to depend on the parameter θ in a different way than given in (27) and (28). The path (see Figure 4) consists of six segments. At the first switching point the lower bound for displacement u_2 and the spring constant k_4 become active. The second segment is terminated when the constraint for displacement u_2 and the spring constant k_4 become inactive. At the next switching point the lower constraint for the displacement u_1 and the spring constant k_4 become active. Later the constraint for displacement u_1 becomes inactive (segment 5). In segment 6 the upper constraint for the displacement u_1 becomes active.

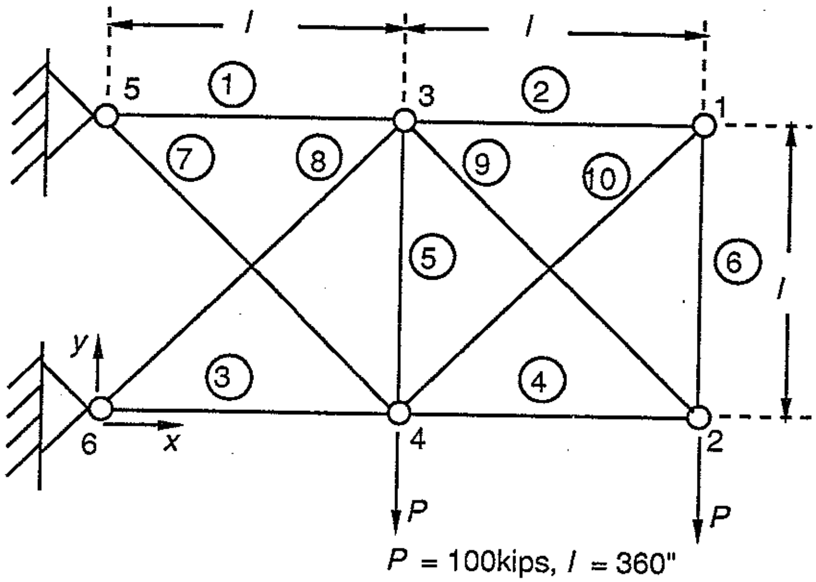

Figure 5. Ten-bar truss configuration.

7. Ten-bar truss example.

7.1. Problem definition.

The ten-bar truss shown in Figure 5 is a well known structural optimization problem. The truss is designed for minimum weight subject to the condition that stresses in bars do not exceed the allowable stresses $\sigma_{mi}(\theta)$, $i = 1, \ldots, 10$.

The problem is formulated as

$$\text{minimize } w(\mathbf{a}) = \sum_{i=1}^{10} \rho l_i a_i, \tag{36}$$

subject to

$$G_i = 0.1 - a_i \leqq 0, \qquad i = 1, \ldots, 10, \tag{37}$$

$$G_j = -\sigma_{mj}(\theta) + \sigma_j(\mathbf{a}, \mathbf{F}) \leqq 0, \qquad j = 11, \ldots, 20, \tag{38}$$

$$G_j = -\sigma_{mj}(\theta) - \sigma_j(\mathbf{a}, \mathbf{F}) \leqq 0, \qquad j = 21, \ldots, 30, \tag{39}$$

where $\mathbf{F}$ is the force vector, $\mathbf{l}$ is a bar-length vector and $\mathbf{a}$ is the design variable vector consisting of cross-sectional areas. The solution needs to be obtained for $\theta \in (\theta_a, \theta_b)$, where θ is the homotopy parameter described later.

The Lagrangian function and Kuhn-Tucker conditions for this problem are:

$$L = \sum_{i=1}^{10} \rho l_i a_i + \sum_{i=1}^{10} \lambda_i (0.1 - a_i) + \sum_{j=11}^{20} \lambda_j (\sigma_{j-10} - \sigma_{m(j-10)}) + \sum_{l=21}^{30} \lambda_l (-\sigma_{l-20} - \sigma_{m(l-20)}), \tag{40}$$

$$\rho l_i + \sum_{j=11}^{20} \lambda_j \frac{\partial \sigma_{j-10}}{\partial a_i} - \sum_{j=21}^{30} \lambda_j \frac{\partial \sigma_{j-20}}{\partial a_i} - \lambda_i = 0, \qquad i = 1, \ldots, 10, \tag{41}$$

$$G_j \lambda_j = 0, \qquad j = 1, \ldots, 30, \tag{42}$$

$$\lambda_j \geqq 0, \qquad j = 1, \ldots, 30, \tag{43}$$

$$G_j \leqq 0, \qquad j = 1, \ldots, 30. \tag{44}$$

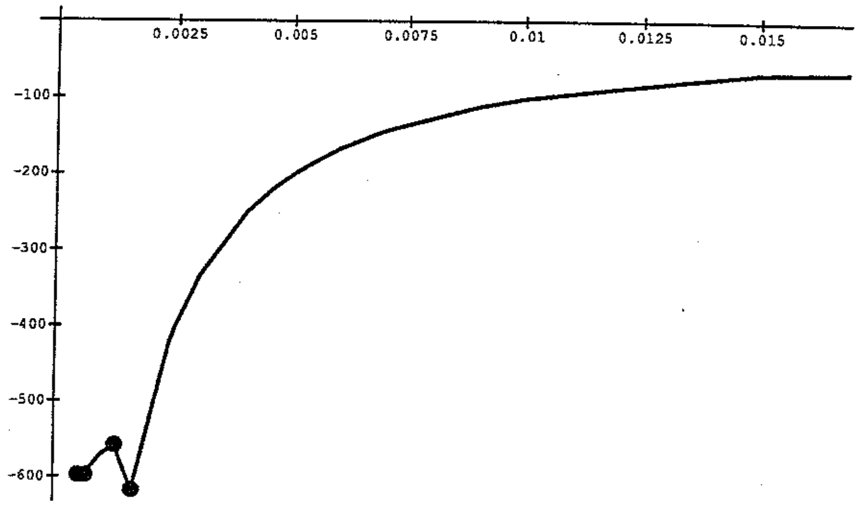

Figure 6. Stress $\sigma_4(\theta)$ from Table 8.

7.2. Results.

First all the allowable stresses for all bars were set equal in magnitude to the reciprocal of the homotopy parameter θ (that θ is the reciprocal of the magnitude of the stress allowable). The results are presented in Table 8 and Figure 6.

For small values of θ the allowable stress is very large and all the design variables are assumed to be at their lower bound ($a_0 = 0.1 \text{ in}^2$). When θ exceeds the reciprocal of the greatest stress magnitude for this minimum gage design some areas must increase. The value of θ is increased and the optimal design variables and the Lagrange multipliers associated with active constraints for stresses are computed. The reason for terminating each segment is given in the table. The classical solution for this problem is obtained when the allowable stress is 25 ksi ($\theta = 0.04$).

In Table 9 the allowable stresses for bars $1, \ldots, 8, 10$ are fixed and equal to 25 ksi whereas the allowable stress for bar 9 is an increasing function of θ ($-21 - 100\theta \leq \sigma_{m9} \leq 21 + 100\theta$). It is known (e.g., [8]) that when the allowable stress is larger than 37.5 ksi the optimal design is no longer fully stressed, as member 9 is neither at the allowable maximum stress nor at minimum gage. The first segment starts at the optimum point for all allowable stresses equal to 25 ksi. Next the allowable stress for bar 9 is increased. The path of optimal points consists of three segments. For $\theta = 0.09177669$ ($\sigma_{m9} = 30.17$ksi) the cross-sectional area of bar 10 becomes an active design variable and the stress in that bar assumes the maximum allowable value (constraint on σ_{10} becomes active). For $\theta = 0.16500000$ ($\sigma_9 = 37.5$ ksi) design variables a_2 and a_6 become active, stresses in these bars attain the maximum allowable value (constraints for σ_2 and σ_6 become active), and the constraint for the stress in bar 9 becomes inactive. For increasing values of θ all design variables and all the Lagrange multipliers remain at the same value.

8. Concluding remarks.

An algorithm was developed for tracking paths of optimal solutions of inequality constrained nonlinear programming problems as a function of a parameter. The algorithm employs homotopy zero curve tracking methodology to track segments where the set of active constraints is unchanged. The transition between segments is handled by considering all possible sets of active constraints and eliminating nonoptimal ones based on the signs of the Lagrange multipliers and the derivatives of the optimal solutions with respect to the parameter. The algorithm was validated for various kinds of transitions between segments using a simple spring problem, and was also successfully applied to a well known 10-bar truss problem.

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Appendix. Pseudo-code for algorithm.

The subroutine STEPNF from HOMPACK is used to track the zero curve Γ of the system of equations (5)–(8) in (θ, k, λ) space. The subroutine takes one step at a time along Γ, choosing the optimal size of the step.

A switching point is localized using Hermite cubic interpolation and the secant method (subroutine ROOTNF in HOMPACK). The accuracy of tracking the zero curve Γ and of finding the switching point is set to 10^{-4}.

The variables used by the program are:

LPAR: identity vector of homotopy variables, **IPAR, IVAR:** work identity vectors of homotopy variables, **y:** vector of values of homotopy variables, **w:** work vector of values of homotopy variables, **y**(1): value of the homotopy parameter θ, θ_1: temporary value of the homotopy parameter θ, *flag*: a flag set true for a switching point;

1. $flag := true$; $\theta := \theta_a$; **y**:= solution at θ_a.
2. If $flag = true$ then
    3. set initial values for STEPNF;

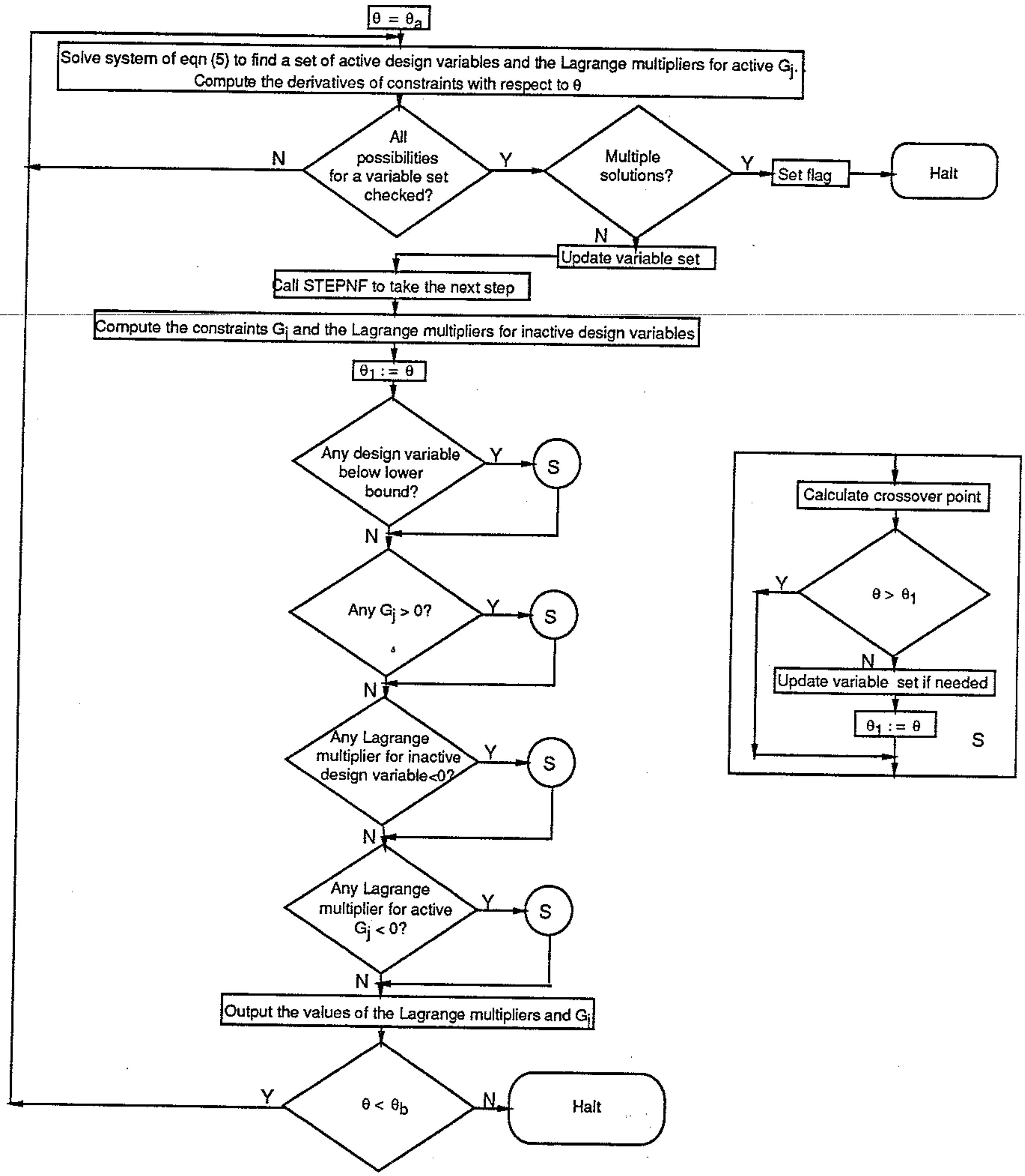

Figure 7. Flowchart for algorithm.

4. do 5.–7. over all possible variable sets:
5. solve the system of equations (5) to check which design variables become inactive and to find the values of the Lagrange multipliers for active constraints of the form (3);
6. if a set of active design variables and active constraints (3) with positive Lagrange multipliers is found, solve the system of equations (22)–(23) to check that no active constraint will be violated for increasing s;
7. if any active constraint will be violated for increasing s go to 5 to find the next set;
8. if there are multiple valid subsets of variable, set a flag and halt;
9. if no set of active design variables and active constraints (3) can be found for the next segment set an error flag and terminate the computations;
10. set initial values for the new variables;
11. $flag := false$; endif.

12. Call STEPNF to take the next step (calculating the new set of variables).
13. Compute the constraints (3) and the Lagrange multipliers for inactive design variables.
14. Save the current vector of variables in **LPAR.**
15. **IVAR:=LPAR.**
16. **w:=y**, θ_1**:=y(1).**
17. If any design variable became less than k_{0i} then
    18. **IPAR:=LPAR;**
    19. choose the design variable with largest violation of its lower bound;
    20. use ROOTNF to find the point where that design variable is equal to k_{0i};
    21. if any other design variable is less than k_{0i} go to 19;
    22. $flag := true$;
    23. **w:=y;**
    24. **IVAR:=IPAR;**
    25. use the current values of the design variables to calculate the G_j's of the form (3);
    26. use the current values of the Lagrange multipliers for active constraints of the form (3) to find the Lagrange multipliers for inactive design variables from equations (5);
    27. θ_1:=y(1); endif.
28. If the value of any G_j of the form (3) becomes greater than 0 then
    29. **IPAR:=LPAR;**
    30. choose the greatest G_j of the form (3) with inactive constraint;
    31. use ROOTNF to find the point where the value of this G_j is equal to 0;
    32. if any G_j of the form (3) with inactive constraint is greater than 0 go to 30;
    33. if $\theta_1 <$ y(1) go to 43;
    34. if $\theta_1 =$ y(1) **IPAR:=IVAR;**
    35. add the Lagrange multiplier associated with the constraint for G_j to the set of variables **IPAR;**
    36. if any other G_j with inactive constraint of the form (3) is equal to 0 add the Lagrange multipliers associated with this constraint to the set of variables **IPAR;**
    37. $flag := true$;
    38. **w:=y;**
    39. **IVAR:=IPAR;**
    40. use the current values of the design variables to calculate the G_j's of the form (3);
    41. use the current values of the Lagrange multipliers for active constraints (3) to find the Lagrange multipliers for inactive design variables from equations (5);

42. θ_1:=y(1); endif.

43. if any Lagrange multiplier for inactive design variable is less than 0 then
    44. **IPAR:=LPAR;**
    45. choose the most negative Lagrange multipliers associated with an inactive design variable;
    46. use ROOTNF to find the point where the Lagrange multiplier is equal to 0;
    47. if any other Lagrange multipliers associated with inactive design variables are negative go to 45;
    48. if $\theta_1 < y(1)$ go to 55;
    49. if $\theta_1 = y(1)$ **IPAR:=IVAR;**
    50. *flag* := *true*;
    51. w:=y
    52. **IVAR:=IPAR**
    53. use the current values of the design variables to calculate the G_j's of the form (3);
    54. use the current values of the Lagrange multipliers for active constraints of the form (3) to find the values of the Lagrange multipliers for inactive design variables from equations (5); endif.

55. if any Lagrange multiplier for active constraint of the form (3) is less than 0 then
    56. **IPAR:=LPAR;**
    57. choose the most negative Lagrange multiplier associated with an active constraint of the form (3);
    58. use ROOTNF to find the point where the Lagrange multiplier is equal to 0;
    59. if any other Lagrange multipliers associated with active constraints of the form (3) are negative go to 57;
    60. if $\theta_1 < y(1)$ go to 69;
    61. if $\theta_1 = y(1)$ **IPAR:=IVAR;**
    62. remove the Lagrange multiplier from the set of variables;
    63. if any other Lagrange multipliers associated with active constraints of the form (3) are equal to 0 remove these Lagrange multipliers from the set of variables;
    64. *flag* := *true*;
    65. w:=y
    66. **IVAR:=IPAR**
    67. use the current values of the design variables to calculate the G_j's of the form (3);
    68. use the current values of the Lagrange multipliers for active constraints of the form (3) to find the values of the Lagrange multipliers for inactive design variables from equations (5); endif.

69. **LPAR:=IVAR.**
70. y:=w.
71. Output the values of the Lagrange multipliers, G_j's of the form (3) and design variables.
72. If $\theta < \theta_b$ then go to 2 else halt.

Table 1. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5$.

Forces: $F_1 = 100$, $F_2 = -300$, $F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	θ	0.00727	0.00768	u_3	137.50	130.10
	k_1	1.00000	1.39271	u_2	−50.00	−59.85
	λ_8	0.04266	0.06618	u_1	62.50	50.18
	Lagrange multiplier for k_5 lower bound becomes equal to 0					
2.	θ	0.00768	0.01118	u_3	130.10	89.43
	k_1	1.39271	1.15654	u_2	−59.85	−89.43
	λ_8	0.06618	0.13861	u_1	50.18	31.68
	k_5	1.00000	1.82648			
	Constraint on u_2 becomes active					
3.	θ	0.01118	0.01125	u_3	89.43	88.88
	k_1	1.15654	1.00000	u_2	−89.43	−88.88
	λ_8	0.13861	0.15421	u_1	31.68	33.33
	k_5	1.82648	1.87500			
	λ_{10}	0.00000	0.01875			
	Lagrange multiplier for k_4 lower bound becomes equal to 0					
4.	θ	0.01125	0.08000	u_3	88.88	12.50
	λ_8	0.15421	6.08000	u_2	−88.88	−12.50
	k_5	1.87500	10.00000	u_1	33.33	12.50
	λ_{10}	0.01875	1.60000			
	k_4	1.00000	11.00000			
	k_1	1.00000	6.00000			
	Constraint on u_1 becomes active					
5.	θ	0.08000	0.11764	u_3	12.50	8.49
	λ_8	6.08000	12.48470	u_2	−12.50	−8.49
	k_5	10.00000	13.76484	u_1	12.50	8.49
	λ_{10}	1.60000	3.68175			
	k_4	11.00000	16.64726			
	k_1	6.00000	9.76484			
	λ_6	0.00000	0.44292			

All Lagrange multipliers are positive. They increase or remain at the same values.

Table 2. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + k_5$.

Forces: $F_1 = 100$, $F_2 = -300$, $F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	θ	0.00727	0.01125	u_3	137.50	88.88
	k_5	1.00000	1.87500	u_2	−50.00	−88.88
	λ_8	0.01163	0.02784	u_1	62.50	33.33
Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_2 becomes active						
2.	θ	0.01125	0.03000	u_3	88.88	33.33
	k_5	1.87500	5.00000	u_2	−88.88	−33.33
	λ_8	0.05484	0.38999	u_1	33.33	33.33
	λ_{10}	0.03374	0.23999			
	k_4	1.00000	3.50000			
Lagrange multiplier for k_2 lower bound becomes equal to 0 and constraint on u_1 becomes active						
3.	θ	0.03000	0.14410	u_3	33.33	6.93
	k_5	5.00000	27.82023	u_2	−33.33	−6.93
	λ_8	0.38999	8.45016	u_1	33.33	6.93
	λ_{10}	0.23999	5.26333			
	k_4	3.50000	14.91011			
	k_2	1.00000	6.70505			
	λ_6	0.00000	0.82210			

The Lagrange multipliers associated with inactive spring constants remain at the same values ($\lambda_1 = 1.0$, $\lambda_3 = 3.0$).

Table 3. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 1.75734k_4 + 5k_5$.

Forces: $F_1 = 100,\ F_2 = -300,\ F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	θ	0.00727	0.00768	u_3	137.50	130.10
	k_1	1.00000	1.39271	u_2	−50.00	−59.85
	λ_8	0.00426	0.06618	u_1	62.50	50.18
Lagrange multipliers for k_4 and k_5 lower bounds become equal to 0 simultaneously						
2.	θ	0.00768	0.01992	u_3	130.10	50.17
	λ_8	0.06618	0.38823	u_2	−59.85	−50.17
	k_1	1.39271	2.26352	u_1	50.18	23.45
	k_4	1.00000	2.25560			
	k_5	1.00000	2.92772			
Constraint on u_2 becomes active						
3.	θ	0.01992	0.09121	u_3	50.17	10.96
	λ_8	0.38823	6.45331	u_2	−50.17	−10.96
	λ_{10}	0.00000	0.73171	u_1	23.45	10.96
	k_1	2.26352	7.12131			
	k_4	2.25560	12.68197			
	k_5	2.92772	11.12131			
Constraint on u_1 becomes active						
4.	θ	0.09121	0.12734	u_3	10.96	7.85
	λ_8	6.45331	11.91582	u_2	−10.96	−7.85
	λ_{10}	0.73171	1.62785	u_1	10.96	7.85
	λ_6	0.00000	0.46004			
	k_1	7.12131	10.73401			
	k_4	12.68197	18.10101			
	k_5	11.12131	14.73401			

The Lagrange multipliers associated with the active spring constants remain at the same values or increase ($\lambda_3 = 3.0,\ \lambda_2 = 8.24$).

Table 4. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5$.

Forces: $F_1 = 100$, $F_2 = -133.33$, $F_3 = 100$.

segment	variable	start value	end value	displacement	start value	end value
1.	θ	0.03000	0.40465	u_3	33.33	2.47
	k_1	1.00000	25.97681	u_2	−33.33	−2.47
	λ_8	0.15000	17.18364	u_1	33.33	2.47
	λ_{10}	0.09000	18.90107			
	λ_6	0.03000	13.03830			
	k_4	1.00000	19.73261			
	k_2	1.00000	7.24420			

The values of the Lagrange multipliers for inactive spring constants k_3 and k_5 change very slowly. ($\lambda_3 = 3.0$, $\lambda_5 = 3.0$)

Table 5. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5$.

Forces: F_1=103.33, F_2=−133.33, F_3=100.

segment	variable	start value	end value	displacement	start value	end value
1.	θ	0.02823	0.03000	u_3	34.58	33.33
	k_1	1.00000	1.10000	u_2	−31.66	−33.33
	λ_6	0.04517	0.05100	u_1	35.41	33.33
Lagrange multipliers for k_4 and k_2 lower bonds become equal to 0, constraints on u_3 and u_2 become active						
2.	θ	0.03000	0.39936	u_3	33.33	2.50
	λ_8	0.14999	16.74794	u_2	−33.33	−2.50
	λ_{10}	0.09000	18.40774	u_1	33.33	2.50
	λ_6	0.03299	13.03830			
	k_1	1.10000	26.95553			
	k_2	1.00000	7.15607			
	k_4	1.00000	19.46823			

The values of the Lagrange multipliers for inactive spring constants k_3 and k_5 change very slowly ($\lambda_3 = 3.0$, $\lambda_5 = 3.0$).

Table 6. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + (2 - e^{(\theta/0.01171 - 1)})k_5$.

Forces: $F_1 = 100$, $F_2 = -300$, $F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	θ	0.00727	0.01125	u_3	137.50	88.88
	k_5	1.00000	1.87500	u_2	−50.00	−88.88
	λ_8	0.01530	0.02893	u_1	62.50	33.33
Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_2 becomes active						
2.	θ	0.01125	0.01818	u_3	88.88	55.00
	k_5	1.87500	3.03011	u_2	−88.88	−55.00
	λ_8	0.05581	0.09823	u_1	33.33	33.33
	λ_{10}	0.03360	0.09260			
	k_4	1.00000	1.92409			
Lagrange multiplier for k_2 lower bound becomes equal to 0						
3.	θ	0.01818	0.02540	u_3	55.00	39.35
	k_5	3.03011	4.62390	u_2	−55.00	−39.35
	λ_8	0.09823	0.00000	u_1	33.33	18.00
	λ_{10}	0.09260	0.18975			
	k_4	1.92409	2.49822			
	k_2	1.00000	1.80146			

For $\theta = 0.02540$ the path terminates with no neighbouring solutions. The problem becomes unbounded ($c(k) \to -\infty$).

Table 7. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + k_5$.

Forces: $F_1 = 100, \quad F_2 = -300, \quad F_3 = 400.$

$$G_7 = u_2 - 1/\theta - 0.009(1/\theta - 137.5)^2,$$

$$G_{10} = -u_2 - 1/\theta - 0.009(1/\theta - 137.5)^2.$$

segment	variable	start value	end value	displacement	start value	end value
1.	θ	0.00727	0.01846	u_3	137.50	54.16
	k_5	1.00000	3.46153	u_2	−50.00	−116.66
	λ_8	0.01163	0.07498	u_1	62.50	12.50
Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_2 becomes active						
2.	θ	0.01846	0.04800	u_3	54.16	20.83
	k_5	3.46153	9.96000	u_2	−116.66	−143.33
	λ_8	0.10127	0.52656	u_1	12.50	−7.50
	λ_{10}	0.03287	0.02460			
	k_4	1.00000	1.00000			
k_4 and the constraint for u_2 become inactive						
3.	θ	0.04800	0.08000	u_3	20.83	12.50
	k_5	9.96000	17.00000	u_2	−143.33	−150.00
	λ_8	0.50688	1.40800	u_1	−7.50	−12.50
Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_1 becomes active						
4.	θ	0.08000	0.21000	u_3	12.50	4.76
	k_5	17.00000	43.00000	u_2	−150.00	−119.04
	k_4	1.00000	1.50000	u_1	−12.50	−4.76
	λ_8	1.43384	9.15923			
	λ_9	0.04307	0.00000			
Constraint on u_1 becomes inactive						
5.	θ	0.21000	0.34088	u_3	4.76	2.93
	k_5	43.00000	67.17601	u_2	−119.04	−94.13
	λ_8	9.15923	23.11248	u_1	−4.76	2.93
	k_4	1.50000	2.09066			
Constraint on u_1 becomes active						
6.	θ	0.34088	0.60946	u_3	2.93	1.64
	k_5	67.17601	120.89250	u_2	−94.13	−96.71
	λ_8	23.11248	73.88685	u_1	2.93	1.64
	k_4	2.09066	2.05004			
	λ_6	0.00000	0.27801			

The Lagrange multipliers associated with inactive spring constants k_1, k_2, k_3 change very slowly.

Table 8. Ten-bar truss example with uniform stress limits:
$-1/\theta \le \sigma_i \le 1/\theta, \quad i = 1, \ldots, 10$, areas given in in^2.

segment	variable	start value	end value	stress (ksi)	start value	end value
1.	θ	0.00048	0.00051	σ_1	1953.65	1939.50
	a_3	0.10000	0.10623	σ_2	401.25	402.71
	λ_{23}	0.00005	0.00006	σ_3	−2046.35	−1939.50
				σ_4	−598.75	−597.29
				σ_5	354.90	342.21
				σ_6	401.25	402.71
				σ_7	1479.76	1499.76
				σ_8	1348.67	1328.65
				σ_9	846.77	844.68
				σ_{10}	567.45	569.51
Lagrange multiplier for a_1 lower bound becomes equal to 0, constraint on σ_1 becomes active						
2.	θ	0.00051	0.00066	σ_1	1939.50	1499.76
	a_1	0.10000	0.12932	σ_2	402.71	402.71
	a_3	0.10623	0.13738	σ_3	−1939.50	−1499.76
	λ_{23}	0.00005	0.00009	σ_4	−597.29	−597.28
	λ_{11}	0.00051	0.00008	σ_5	342.21	342.21
				σ_6	402.71	402.71
				σ_7	1499.76	1499.76
				σ_8	1328.65	1328.65
				σ_9	844.68	844.68
				σ_{10}	569.51	569.51
Lagrange multiplier for a_7 lower bound becomes equal to 0, constraint on σ_7 becomes active						
3.	θ	0.00066	0.00126	σ_1	1499.76	788.78
	a_1	0.12932	0.19748	σ_2	402.71	442.24
	a_3	0.13738	0.30961	σ_3	−1499.76	−788.79
	a_7	0.10000	0.25857	σ_4	−597.28	−557.76
	λ_{23}	0.00012	0.00046	σ_5	342.21	0.00
	λ_{17}	0.00016	0.00059	σ_6	402.71	442.24
	λ_{11}	0.00005	0.00018	σ_7	1499.76	788.79
				σ_8	1328.65	788.79
				σ_9	844.68	788.79
				σ_{10}	569.51	625.43
Lagrange multipliers for a_8, a_9 lower bounds become equal to 0, constraints on σ_8, σ_9 become active						
4.	θ	0.00126	0.00162	σ_1	788.78	616.78
	a_1	0.19748	0.26213	σ_2	442.24	383.22
	a_3	0.30961	0.38639	σ_3	−788.79	−616.78
	a_7	0.25857	0.31715	σ_4	−557.76	−616.78

a_8	0.10000	0.14142	σ_5	0.00	0.00
a_9	0.10000	0.14142	σ_6	442.24	383.22
λ_{23}	0.00034	0.00057	σ_7	788.79	616.78
λ_{19}	0.00026	0.00043	σ_8	788.79	616.78
λ_{18}	0.00026	0.00043	σ_9	788.79	616.78
λ_{17}	0.00037	0.00061	σ_{10}	625.43	542.95
λ_{11}	0.00029	0.00048			

Lagrange multiplier for a_4 lower bound becomes equal to 0, constraint on σ_4 becomes active

5.

θ	0.00162	0.06266	σ_1	616.78	15.96
a_1	0.26213	12.47049	σ_2	383.22	9.92
a_3	0.38639	12.59476	σ_3	−616.78	−15.96
a_4	0.10000	6.20418	σ_4	−616.78	−15.96
a_7	0.31715	8.94977	σ_5	0.00	0.0
a_8	0.14142	8.77403	σ_6	383.22	9.92
a_9	0.14142	8.77403	σ_7	616.78	15.96
λ_{23}	0.00057	0.78533	σ_8	616.78	15.96
λ_{19}	0.00043	0.78533	σ_9	616.78	15.96
λ_{18}	0.00043	0.78533	σ_{10}	542.95	14.02
λ_{17}	0.00061	0.78533			
λ_{11}	0.00048	0.78533			
λ_{24}	0.00000	0.39266			

Table 9. Ten-bar truss example with variable allowable stress for member 9:
$-25.0 \leq \sigma_i \leq 25.0, \qquad i = 1, \ldots, 8, 10,$
$-25.0 - 100\theta \leq \sigma_9 \leq 21.0 + 100\theta$, areas given in in^2.

segment	variable	start value	end value	stress (ksi)	start value	end value
1.	θ	0.04	0.09177	σ_1	25.00	25.00
	a_1	7.94000	7.93000	σ_2	15.52	17.67
	a_3	8.06000	8.07071	σ_3	−25.00	−25.00
	a_4	3.94000	3.93000	σ_4	−25.00	−25.00
	a_7	5.74000	5.75685	σ_5	0.05	0.0
	a_8	5.57000	5.55685	σ_6	15.52	17.67
	a_9	5.57000	4.60344	σ_7	25.00	25.00
	λ_{23}	0.31357	0.32062	σ_8	25.00	25.00
	λ_{19}	0.31357	0.22013	σ_9	25.00	30.17
	λ_{18}	0.31357	0.31874	σ_{10}	21.95	25.00
	λ_{17}	0.31357	0.32125			
	λ_{11}	0.31357	0.31937			
	λ_{24}	0.15678	0.15937			

Lagrange multipliers for a_{10} lower bound becomes equal to 0, constraint on σ_{10} becomes active

segment	variable	start value	end value	stress (ksi)	start value	end value
2.	θ	0.09177	0.16500	σ_1	25.00	25.00
	a_1	7.93000	7.90000	σ_2	17.67	25.00
	a_3	8.07071	8.10000	σ_3	−25.00	−25.00
	a_4	3.93000	3.90000	σ_4	−25.00	−25.00
	a_7	5.75685	5.79827	σ_5	0.0	0.0
	a_8	5.55685	5.51543	σ_6	17.67	25.00
	a_9	4.60344	3.67695	σ_7	25.00	25.00
	a_{10}	0.10000	0.14213	σ_8	25.00	25.00
	λ_{23}	0.31400	0.32333	σ_9	30.17	37.50
	λ_{19}	0.21300	0.13999	σ_{10}	25.00	25.00
	λ_{18}	0.31000	0.31333			
	λ_{17}	0.31600	0.32666			
	λ_{11}	0.31100	0.31666			
	λ_{24}	0.15500	0.15666			
	λ_{20}	0.00290	0.00666			

Lagrange multipliers for a_6, a_2 bounds become equal to 0, constraints on σ_6, σ_2 become active, constraint on σ_9 becomes inactive

segment	variable	start value	end value	stress (ksi)	start value	end value
3.	θ	0.16500	15.96428	σ_1	25.00	25.00
	a_1	7.90000	7.90000	σ_2	25.00	25.00
	a_3	8.10000	8.10000	σ_3	−25.00	−25.00
	a_4	3.90000	3.90000	σ_4	−25.00	−25.00
	a_7	5.79827	5.79827	σ_5	0.0	0.0
	a_8	5.51543	5.51543	σ_6	25.00	25.00

a_9	3.67695	3.67695
a_{10}	0.14213	0.14213
λ_{23}	0.39333	0.39333
λ_{20}	0.14666	0.14666
λ_{18}	0.17333	0.17333
λ_{17}	0.46666	0.46666
λ_{11}	0.24666	0.24666
λ_{24}	0.08666	0.08666
λ_{16}	0.07333	0.08666
λ_{12}	0.06666	0.06666
a_6	0.10000	0.10000

σ_7	25.00	25.00
σ_8	25.00	25.00
σ_9	37.50	1617.40
σ_{10}	25.00	25.00