New Homotopy Solution Techniques Applied to Variable Geometry Trusses

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NEW HOMOTOPY SOLUTION TECHNIQUES APPLIED TO VARIABLE GEOMETRY TRUSSES

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ABSTRACT

A VGT, or Variable Geometry Truss, can be thought of as a statically determinate truss that has been modified to contain some number of variable length members. These extensible members allow the truss to vary its configuration in a controlled manner. Some of the typical applications envisioned for VGTs are as booms to position equipment in space, as supports for space antennae, and as berthing devices. Recently, they have also been proposed as parallel-actuated, long chain, high dexterity manipulators. This paper will demonstrate the use of homotopy continuation in solving the kinematics of relatively complex variable geometry trusses (VGTs) including the octahedron and the decahedron. The procedural aspects are described in detail with the help of examples. Results of the example problems are also presented.

INTRODUCTION

All VGTs are composed of some combination of fundamental units, referred to as unit cells. These unit cells, or basic cells, form the building blocks for the analysis and design of all trusses. The necessary and sufficient rules for determining admissable VGT unit cells are enumerated in detail in the paper by Arun, Reinholtz and Watson (1990).

The objective of this work is to find all possible assemblies of any VGT unit cell given its link dimensions. This will be referred to as the forward kinematics problem. Problems of this type lead to systems of polynomial equations. Commonly, such systems are solved by an iterative numerical method, usually the Newton-Raphson method. The shortcoming of the n-dimensional Newton-Raphson method is that excellent initial guesses are required to ensure convergence. Success is never guaranteed because there is no sure way of making a good initial guess. The method also runs into problems in finding and distinguishing multiple solutions.

On the other hand, recently developed methods in homotopy continuation for polynomials (polynomial continuation) are not only global, but also exhaustive; i. e., they do not require good initial guesses and also guarantee convergence to all solutions. Homotopies are a traditional part of topology and only recently have begun to be used for practical numerical computation.

The idea of polynomial continuation was first suggested by Garcia and Zangwill (1977) and Drexler (1977). Later Chow, Mallet-Paret, and Yorke (1979) presented an important refinement to the Garcia and Zangwill work. A number of papers extending the idea have followed: Brunovsky and Meravy (1984), Drexler (1978), Garcia and Li (1980), Garcia and Zangwill (1979, 1981), Li, Sauer and Yorke (1987), Morgan (1983, 1986, 1987), Morgan and Sommese(1987), Morgan, Sommese, and Watson (1988). A computer program called HOMPACK (Watson, Billups, and Morgan, 1987) implements homotopy algorithms and solves polynomial systems.

The first application of a homotopy algorithm to a kinematics problem was in the analysis of the general 6-R manipulator (Tsai and Morgan, 1985). All sixteen solutions to a system of eight second degree polynomials in eight unknowns were found using homotopy continuation. Issues involved in solving polynomial systems arising in kinematics using polynomial continuation are addressed in a publication by Wampler, Morgan and Sommese (1988). Recent publications on the use of continuation to solve problems in kinematics include (Tsai, Lu, 1989), (Morgan, Wampler, 1989), (Subbian, Flugrad, 1989). The homotopy continuation method and the mathematical theory behind it are described in detail in the above-mentioned references. A summary is presented in a later section of this paper.

The idea behind solving systems of nonlinear equations using continuation is that small changes in the parameters of the system usually produce small changes in the solutions. Suppose the solutions to problem G are known and a similar problem F is to be solved. By tracking the solutions of the problem as the parameters are slowly changed from those of G to those of F, some solutions to problem F will be found. A solution by continuation consists of three elements:

- a start system with known solutions ;
- a schedule for changing the parameters from those of the start system to those of the target system ;
- a method for tracking the solutions as the scheduled transformation proceeds.

Let $f(z) = 0$ denote a system of n polynomial equations in n unknowns with complex coefficients. Generally, such a system has many solutions. Homotopy continuation is used to find all the geometrically isolated solutions of $f(z) = 0$ as follows. Imbed f in a system of n polynomial equations in $n+1$ unknowns where this new system includes the variables of f and a new variable, the homotopy parameter. This new system, the *homotopy*, is denoted by $h(z,t) = 0$ where $h(z,1) = f(z)$ for all z and $h(z,0) = 0$ denotes the start system $g(z) = 0$ whose solutions are known. Thus, for one value of the homotopy parameter, the new system can be satisfactorily solved and for another it is identically equal to $f(z) = 0$. The continuation process attempts to solve $f(z) = 0$ by evolving or "continuing" the full set of known solutions into the full set of solutions to $f(z) = 0$. Thus, the homotopy parameter t varies between 0 and 1, and the solutions of $h(z,0) = 0$ must be continued into the solutions of $h(z,1) = 0$.

When homotopy continuation is used to compute the full solution list for a polynomial system, the continuation is carried out in complex projective space in a complex analytic context. Complicated kinematic problems can be solved routinely using homotopy continuation because of some special properties of polynomials. First, the total number of solutions for the system can be readily calculated. Second, it is often easy to find an appropriate start system with known solutions. Finally, there is a standard formulation for the homotopy from the start system to the target system that guarantees smooth solution paths that do not diverge to infinity, turn back or bifurcate.

SOME KEY CONCEPTS IN HOMOTOPY CONTINUATION

BEZOUT NUMBER

One key concept used in counting the number of solutions to a polynomial system is the ***Bezout number***. Consider a single polynomial equation in one complex variable x:

$$\sum_{j=0}^{n} a_j x^j = 0, \qquad [1]$$

where the a_j's are constants.

This equation has exactly n solutions counting multiplicities. An alternate approach is to define the ***Bezout number*** of a solution as the number of solutions in its neighborhood for small perturbations in the coefficients of the polynomial. For example, the solution x = -1 in $x^2 + 2x + 1 = 0$ has Bezout number 2 because $x^2 + 2x + (1 - \nabla) = 0$ has two solutions $x = -1 \pm \sqrt{\nabla}$. For a single polynomial, the sum of Bezout numbers over all the solutions of an n^{th} degree polynomial is n. This sum is called the *Bezout number of the polynomial*.

For a polynomial in n variables, the degree is the largest degree of any of its terms. The *total degree* of a system of n polynomial equations in n unknowns is the product $\prod_{j=1}^{n} d_j$ where d_j is the degree of the jth equation. Bezout's Theorem states that in general the total number of solutions of such a polynomial system is equal to the total degree of the system. The Bezout number provides an upper bound on the possible number of real and complex solutions.

One important theoretical development in homotopy continuation was the proposal by Wright (1985) and Morgan (1986) that the path-tracking take place in complex projective space P^n rather than real or complex Euclidean space. This is because the structure of the solution set to $f(z)=0$ is generic in P^n. Let $f(z)=0$ denote a polynomial system of n equations in n unknowns with complex coefficients. A homogenization of $f(z)=0$ is defined to be $f'(y)=0$, a system of n homogeneous equations in $n+1$ unknowns ; i. e.,

$$f'_j(y) = y_{n+1}^{d_j}\ f_j\left(\frac{y_1}{y_{n+1}}, \ldots\ldots, \frac{y_n}{y_{n+1}}\right), \qquad j = 1, \ldots, n, \tag{2}$$

where d_j = degree (f_j).

If $f'(y^0)=0$, then $f'(cy^0)=0$ for any complex scalar c. Therefore, solutions of $f'(y)=0$ can be taken to be complex lines through the origin in C^{n+1} (Euclidean space). The set of these lines is called *complex projective space*, P^n, a smooth compact n-complex-dimensional manifold. The solutions to $f'(y)=0$ can be identified with solutions to $f(z)=0$ if $y_{n+1} \neq 0$. All solutions to $f'(y)=0$ with $y_{n+1}=0$ are called *solutions at* infinity. The significance of working in P^n is that if $f'(y)=0$ has a finite number of solutions in P^n, then it has exactly the Bezout number of solutions.

The use of multiple homogeneous variables can sometimes reduce the number of solutions at infinity, which will reduce the computational load when all the solutions of the system are calculated, since the finite solutions are unchanged. The general procedure for m-homogenizing a polynomial system $f(z)=0$ is described below. The variables $\{z_1,\ldots,z_n\}$ are partitioned into m nonempty collections. For notational simplicity, the variables are renamed with double subscripts.

$$\{z_1,\ldots,z_n\} = \bigcup_{j=1}^{m} \{z_{1,j},\ldots, z_{k_j,j}\}, \qquad [3]$$

$$\text{where } \sum_{j=1}^{m} k_j = n.$$

Homogeneous variables $z_{0,j}$ are chosen for $j = 1,\ldots,m$ and hence $Z_j = \{z_{0,j},\ldots,z_{k_j,j}\}$ is defined for $j = 1$ to m. Then the substitution $z_{i,j} \leftarrow \frac{z_{i,j}}{z_{0,j}}$ for $i = 1$ to k_j and $j = 1$ to m is evoked, generating a system $f'(y) = 0$ of n equations in $n + m$ unknowns (once the denominator is cleared of powers of $z_{0,j}$). $f'(y)$ is called *$m-$homogeneous* because the variables are partitioned into m collections, $Z_1, \ldots, Z_m$, so that $f'(y)$ is homogeneous as a system in the variables of any of the collections.

The formula for the Bezout number in this case is as follows. Let the m-homogeneous degree of equation l with respect to group j be d_{jl}. The Bezout number of m-homogeneous polynomial equations is equal to the coefficient of $\prod_{j=1}^{m} \alpha_j^{k_j}$ in the product

$$\prod_{l=1}^{n}\left[\sum_{j=1}^{m} d_{jl}\alpha_j\right] \qquad [4]$$

HOMOTOPY

The schedule for transforming the start system $G(y)$ into the target system $F(y)$ is called a homotopy. The requirement for an acceptable homotopy is that as the transformation proceeds, there should be a constant number of solutions that trace out smooth paths and which are always nonsingular until the target system is reached. One homotopy which suffices is

$$H(y,t) = (1-t)e^{i\theta}G(y) + tF(y), \qquad [5]$$

where θ is picked at random.

START SYSTEM

Three restrictions are placed on a start system:

- All of its solutions must be known.
- Each solution must be nonsingular.
- The start system must have the same m-homogeneous structure (the same d_{jl}, $j = 1,\ldots,m$; $l = 1,\ldots,n$) as the target system.

ANALYSIS OF COMPLEX UNIT CELLS

OCTAHEDRAL VGT

Figure 1a shows a kinematic diagram of one cell of an octahedral VGT. This type of VGT was designed and constructed to test deployment concepts at NASA Langley and is presently being used for research in robotics and vibration control at VPI & SU. This particular design has three extensible actuators connected together to form a triangular frame called the actuator frame. In theory, it is possible to actuate any link of a VGT.

Figure 1b shows an octahedral cell modeled as a ten link mechanism made up of three revolute joints, three prismatic joints and three spheric joints. This mechanism

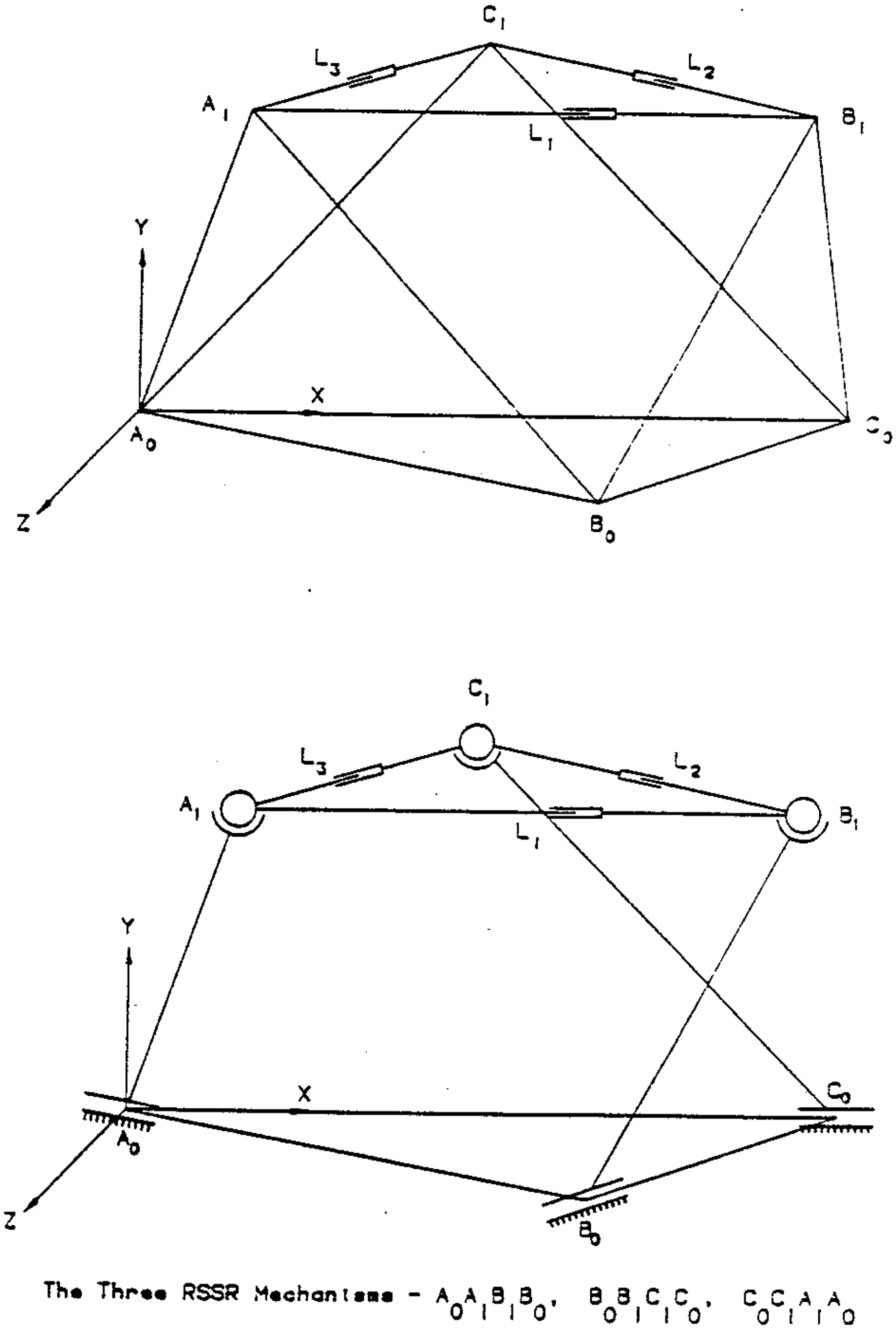

Figure 1. Octahedral VGT and Associated Kinematic Model

can be visualized as three interconnected RSSR mechanisms. The angles $\theta_1, \theta_2, \theta_3$ are the angles made by the links AO-A1, BO-B1, and CO-C1 with the ground.

The forward kinematics problem in this case is to find the three angles $\theta_1, \theta_2, \theta_3$ given the lengths of the twelve variable links that comprise the cell. After considerable algebraic manipulation, the equations relating the lengths(input) and the angles(output) were found to be as follows:

$$L_1^2 - (A\cos\theta_1 + A\cos\theta_2 + B\cos\theta_1\cos\theta_2 - 2B\sin\theta_1\sin\theta_2 + C) = 0,$$

$$L_2^2 - (D\cos\theta_2 + D\cos\theta_3 + E\cos\theta_2\cos\theta_3 - 2E\sin\theta_2\sin\theta_3 + F) = 0,$$

$$L_3^2 - (J\cos\theta_1 + J\cos\theta_3 + K\cos\theta_1\cos\theta_3 - 2K\sin\theta_1\sin\theta_3 + L) = 0. \qquad [6]$$

Since the results of this work were being used in the NASA research projects described earlier, the equations were framed keeping L_1, L_2 and L_3 explicit. Other link dimensions are contained in the coefficients A, B,..., L. Each of these equations is nonlinear and contains the unknown values of θ_i. By making the tangent half angle substitution for sines and cosines, equations [1] can be written as:

$$\alpha_1 t_1^2 + \alpha_2 t_2^2 + \alpha_3 t_1^2 t_2^2 + \alpha_4 t_1 t_2 + \alpha_5 = 0,$$

$$\beta_1 t_2^2 + \beta_2 t_3^2 + \beta_3 t_2^2 t_3^2 + \beta_4 t_2 t_3 + \beta_5 = 0,$$

$$\gamma_1 t_3^2 + \gamma_2 t_1^2 + \gamma_3 t_1^2 t_3^2 + \gamma_4 t_1 t_3 + \gamma_5 = 0. \qquad [7]$$

The coefficients $\alpha_1, \alpha_2, \ldots\ldots\ldots, \gamma_5$ are functions of the link lengths. Each nonlinear equation above is of degree 4 and involves two variables. If one could reduce this system of equations to an equation in one variable, that equation would be of the 64th degree. A closed form solution is unlikely. Even if one were possible, the algebra involved would be complex and tedious. It is far more practical to solve the system of equations numerically.

For the reasons cited earlier, homotopy continuation is used to solve the problem. The system of equations [2] is the target system - the system to be solved. A 3-homogeneous formulation is suggested by the presence of the three output variables. The three groups, each with one member, are: $\{t_1\}, \{t_2\}$, and $\{t_3\}$. The 3-homogeneous Bezout number of the above system of equations is 16. Hence the above system of equations has 16 solutions in the appropriate 3-homogeneous projective space $P^1 \times P^1 \times P^1$.

The 3-homogenization is performed in the following manner. Let

$$t_1 \leftarrow \frac{X}{W_1},$$

$$t_2 \leftarrow \frac{Y}{W_2},$$

$$t_3 \leftarrow \frac{Z}{W_3},$$

where X, Y, Z, W_1, W_2, W_3 are all complex numbers. Making these substitutions in the above equations [2] results in the following 3-homogeneous system :

$$\alpha_1 W_2^2 X^2 + \alpha_2 W_1^2 Y^2 + \alpha_3 W_1^2 W_2^2 + \alpha_4 W_1 W_2 XY + \alpha_5 W_1^2 W_2^2 = 0,$$

$$\beta_1 W_2^2 Z^2 + \beta_2 W_3^2 Y^2 + \beta_3 W_2^2 W_3^2 + \beta_4 W_2 W_3 YZ + \beta_5 W_2^2 W_3^2 = 0,$$

$$\gamma_1 W_1^2 Z^2 + \gamma_2 W_3^2 X^2 + \gamma_3 W_1^2 W_3^2 + \gamma_4 W_1 W_3 XZ + \beta_5 W_2^2 W_3^2 = 0. \qquad [8]$$

As mentioned earlier, the start system must have the same m-homogeneous structure as the target system, and all the solutions must be known. The start system chosen was:

$$(t_1^2 - 4)(t_2^2 - 9) = 0,$$

$$(t_2^2 - 1)(t_3^2 - 4) = 0,$$

$$(t_3^2 - 9)(t_1^2 - 1) = 0. \qquad [9]$$

This system can be homogenized in the same manner. This is just one of the possible start systems. It has exactly 16 solutions, no 3-homogeneous solutions at infinity, and the same degree as equations [8]. Both the target and start systems are homogenized and placed in the homotopy. The path tracking algorithm used to transform solutions of the start system into solutions of the target system is the normal flow algorithm from HOMPACK (Watson, Billups, and Morgan, 1987). The results of an example problem are shown in Figs. 2 and 3. For the example shown, there are 16 possible assemblies for the chosen combination of link lengths. Two views of each assembly are shown, the first row shows the top view, and the second one the corresponding elevation. Closer examination of the figures reveals that eight assemblies are mirror images of the other eight.

Griffis and Duffy (1989) have analyzed the Stewart platform, which is a special case of the Octahedral VGT. Their results are in agreement with the above. One point to note is that they have used an eliminant to reduce the system of equations to one equation in one unknown. The algebra involved is tedious, and involves a lot of bookkeeping. When VGTs like the decahedron are to be analyzed, this approach will be impractical compared to the homotopy continuation approach based on m-homogeneous polynomials.

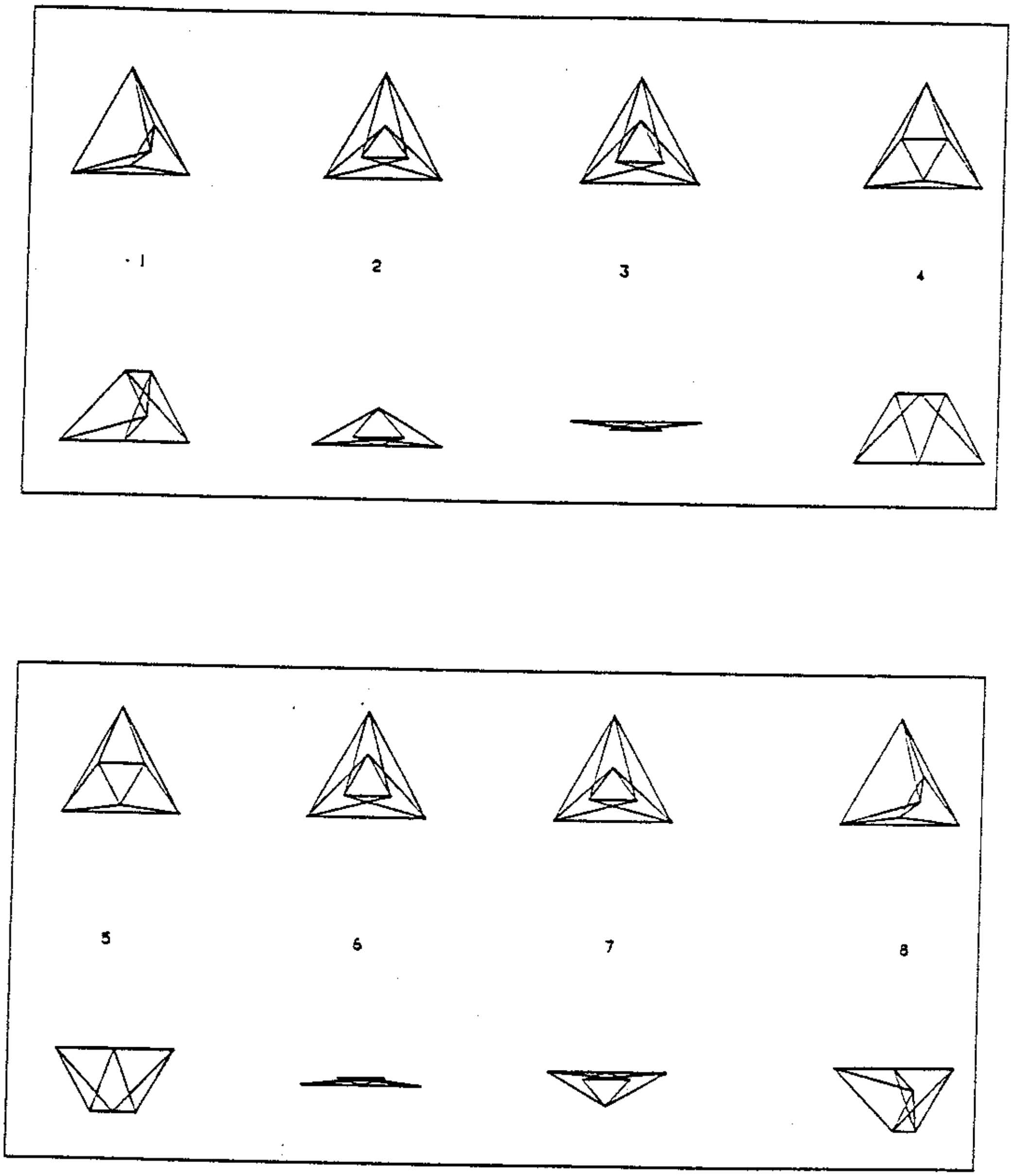

Figure 2. Results - Octahedral VGT Example Problem

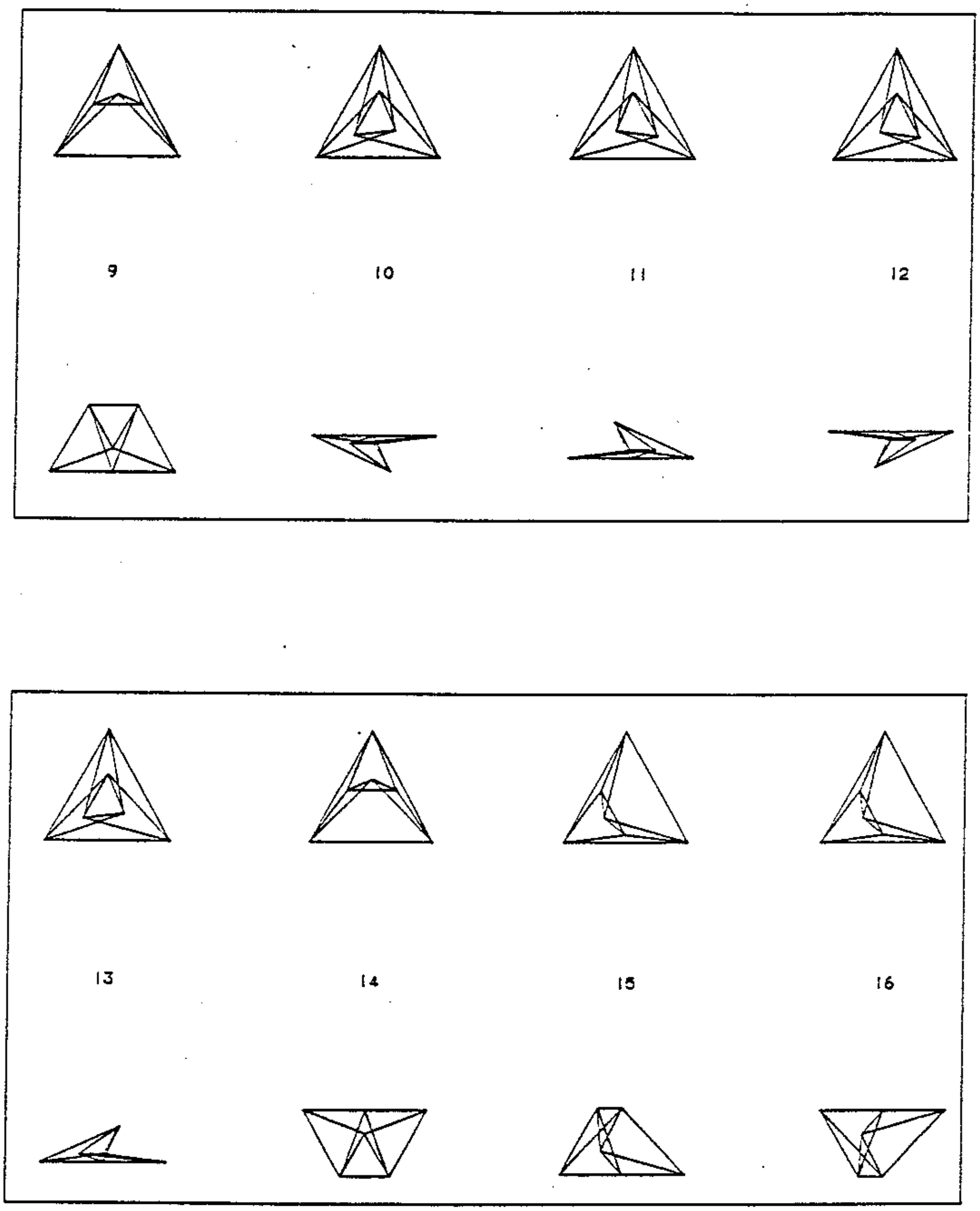

Figure 3. Results - Octahedral VGT Example Problem

DECAHEDRAL VGT

Figure 4 shows a drawing of the decahedron and the kinematic model used for analysis. The forward kinematic problem in this case is to determine the three angles θ_1, θ_2, θ_3, and the unit vector $\overline{u}$ given all the link lengths (refer to Fig. 4).

As can be seen from Fig. 4, there are 2 RSSR mechanisms in the model and five SS dyads. Constraint equations are written for these based on constant dyad length and link rotation. After considerable algebraic manipulation and simplification, the input/output equations achieve the following final form:

$$\alpha_1 t_1^2 + \alpha_2 t_2^2 + \alpha_3 t_1^2 t_2^2 + \alpha_4 t_1 t_2 + \alpha_5 = 0,$$

$$\beta_1 t_2^2 + \beta_2 t_3^2 + \beta_3 t_2^2 t_3^2 + \beta_4 t_2 t_3 + \beta_5 = 0,$$

$$\gamma_1 + \gamma_2 v_x t_1^2 + \gamma_3 v_y t_1^2 + \gamma_4 v_z t_1 + \gamma_5 v_x + \gamma_6 v_y + \gamma_7 t_1^2 = 0,$$

$$\delta_1 + \delta_2 v_x t_2^2 + \delta_3 v_y t_2^2 + \delta_4 v_z t_2 + \delta_2 v_x + \delta_5 v_y + \delta_6 t_2^2 = 0,$$

$$\sigma_1 + \sigma_2 v_x t_3^2 + \sigma_3 v_y t_3^2 + \sigma_4 v_z t_3 + \sigma_5 v_x + \delta_6 v_y + \sigma_7 t_3^2 = 0,$$

$$v_x^2 + v_y^2 + v_z^2 - 1 = 0. \qquad [9]$$

The unknowns in the above system of equations are $t_1, t_2, t_3, v_x, v_y, v_z$. Here, t_1, t_2 and t_3 represent the angles θ_1, θ_2, θ_3 after the tangent half angle substitution and v_x, v_y, v_z are the components of the unit vector along link CG. The coefficients of the terms in the equations are all functions of the link lengths. The total degree of this system of equations is (4) (4) (3) (3) (3) (2) = 864. Such systems are practically impossible to solve by elimination. A numerical solution by a homotopy algorithm on the other hand, is almost routine.

A 4-homogeneous homotopy gives the lowest Bezout number - 48. This implies that the above system does not possess more than 48 solutions. The four groups chosen are $\{t_1,t_2,t_3\},\{v_x\},\{v_y\},\{v_z\}$. The start system used is :

$$(t_1^2-4)(t_2^2-9)=0,$$

$$(t_2^2-1)(t_3^2-16)=0,$$

$$(t_1^2-9)(v_x+2v_y+3v_z-6)=0,$$

$$(t_2^2-4)(2v_x+3v_y+v_z-6)=0,$$

$$(t_3^2-1)(3v_x+v_y+2v_z-6)=0,$$

$$(v_x^2-4)=0. \qquad [10]$$

The start system and the target system are homogenized and the 48 solution paths are tracked using the same normal flow algorithm. The results of an example problem are shown in Figs. 5 and 6. In this example, 20 real solutions were found, i.e., twenty possible assemblies exist for the chosen link lengths. Figures 5 and 6 show the top and front views of the assemblies. Again, ten assemblies are mirror images of the other ten.

GENERAL SOLUTION TECHNIQUES FOR VGT UNIT CELLS

The forward kinematics of the octahedral and decahedral cells were solved using kinematic models. The construction of these models required some intuition and understanding. It is possible to solve the forward kinematics of *any* VGT cell in a most general manner. This is done by writing constraint equations for the links using

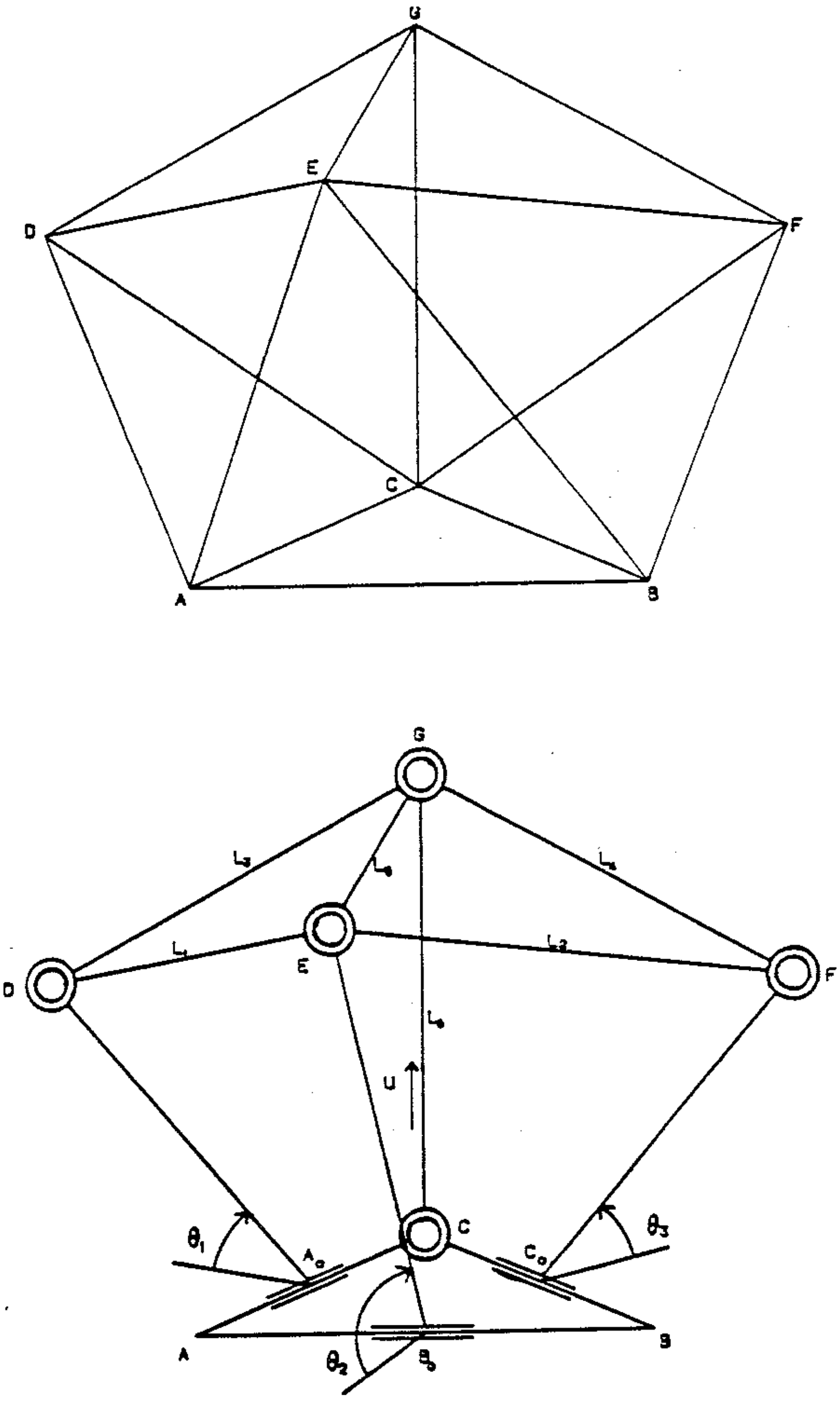

Figure 4. Decahedral VGT and Associated Kinematic Model

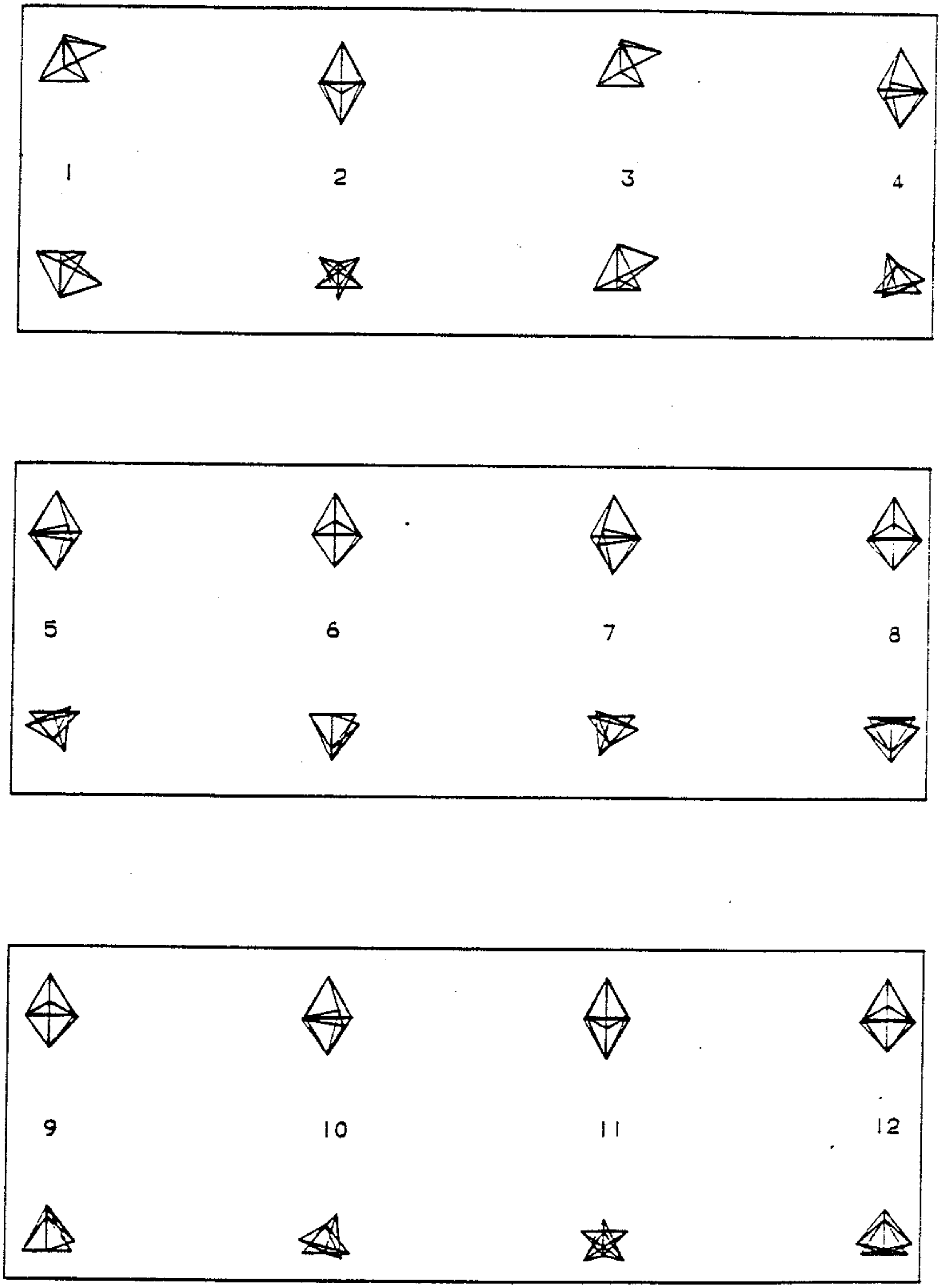

Figure 5. Results - Decahedral VGT Example Problem

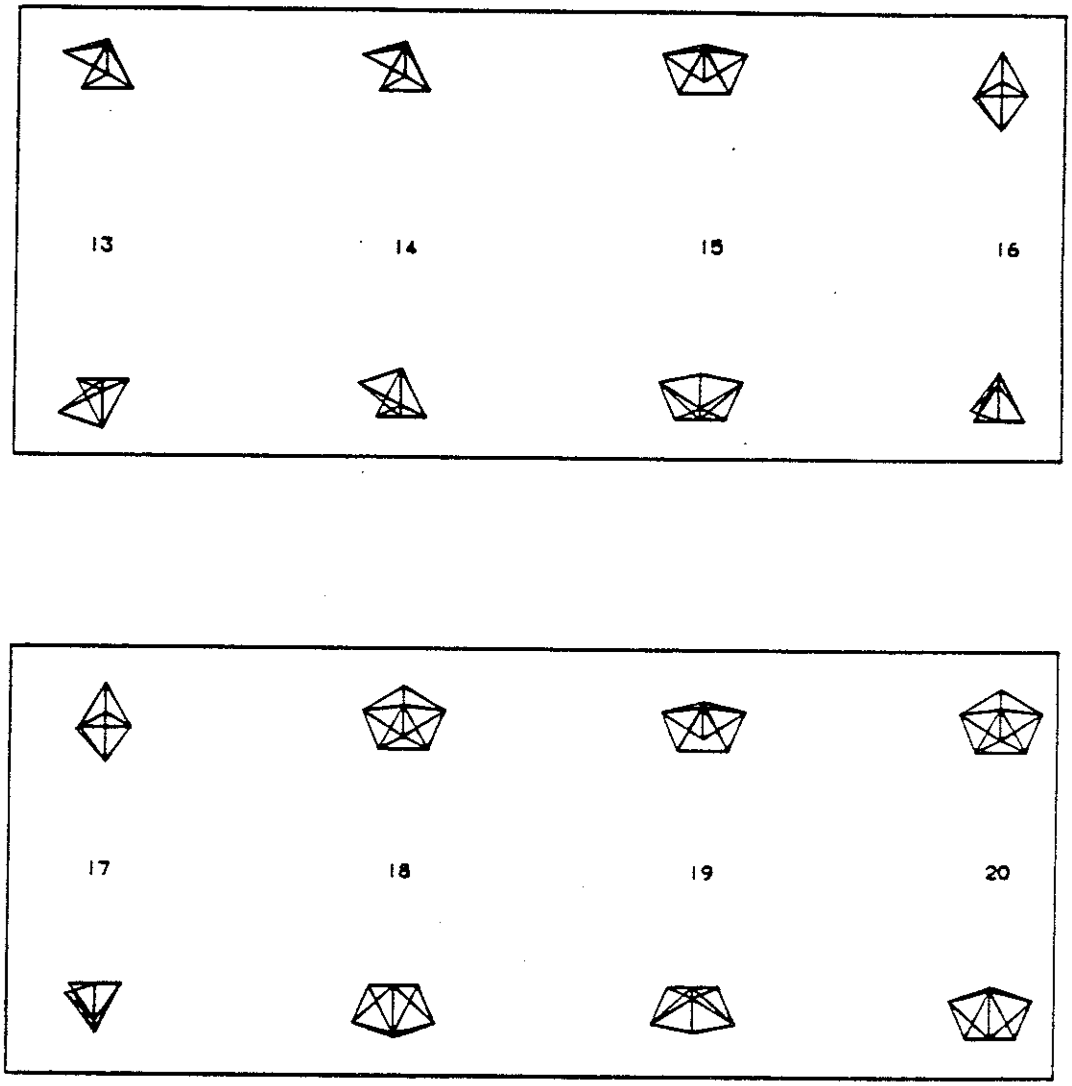

Figure 6. Results - Decahedral VGT Example Problem

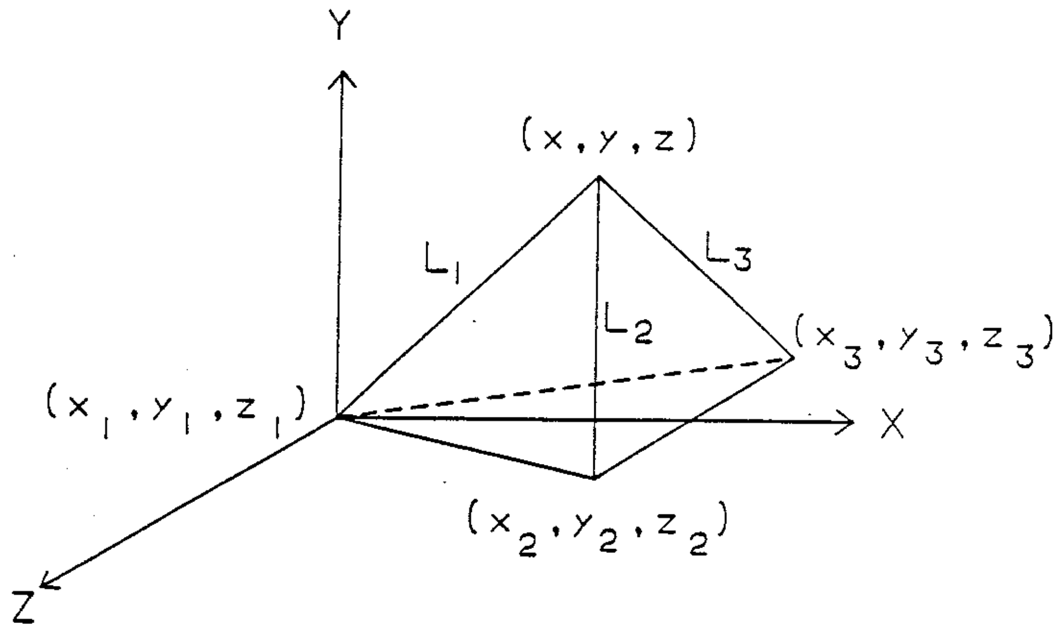

Figure 7. Tetrahedral VGT - General Formulation

the fixed distance between the spheric joints. Using the tetrahedron of Fig. 7 as an example, the constraint equations can be written as:

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=L_1^2,$$

$$(x-x_2)^2+(y-y_2)^2+(z-z_2)^2=L_2^2,$$

$$(x-x_3)^2+(y-y_3)^2+(z-z_3)^2=L_3^2. \qquad [11]$$

The number of such equations for a VGT cell will be equal to the number of links not counting the ground links. Hence, for the octahedral unit cell, there will be 9 such equations, and so on. Such systems of equations can be solved easily using homotopy and all real solutions will be found. However, this particular formulation yields a two-dimensional manifold of solutions at infinity, since this system does not have a finite number of solutions in complex projective space. Thus, the Bezout number is not an useful upper bound on the number of solutions. Hence, using this technique, the forward kinematics of any VGT unit cell can be solved without additional modeling.

CONCLUSIONS

This paper has presented an efficient, robust method for determining all possible assemblies of the most common VGT unit cells. The technique used is homotopy continuation based on the theory of m-homogeneous polynomials. Good initial guesses are not required, and convergence is guaranteed. Such a method is especially useful in the analysis of kinematic structures with a multiplicity of solutions, such as the VGT. As examples, the technique was demonstrated on both the Octahedal and Decahedral VGT unit cells.

REFERENCES

1. Arun, V., Reinholtz, C. F., and Watson, L. T., "Truss-Type Manipulators : A General Method for Enumeration and Analysis ", Submitted to the *Journal of Mechanical Design* , 1990.

2. Brunovsky, P., and Meravy, P., "Solving Systems of Polynomial Equations by Bounded and Real Homotopy", *Numer. Math.* , Vol. 43, 1984, pp. 397-418.

3. Chow, S. N., Mallet-Paret, and Yorke, J. A., "A Homotopy Method for Locating All Zeros of a System of Polynomials", *Functional Differential Equations and Approximation of Fixed Points* , H.-O. Peitgen and H. O. Walther, Editors, Springer Verlag Lecture Notes in Math, 730, New York, 1979, pp. 228-237.

4. Drexler, F. J., "Eine Methode zur Berechung samtlicher Losunger von Polynomgleichungessystemen", *Numer. Math.* , Vol. 29, 1977, pp 45-58.

5. Drexler, F. J., "A Homotopy Method for the Calculation of Zeros of Zero-Dimensional Polynomial Ideals", *In Continuation Methods* , H. G. Wacker (Ed.), Academic Press, New York, 1978, pp 69-93.

6. Duffy, J., and Rooney, J., "A Foundation for a Unified Theory of Analysis of Spatial Mechanisms", Trans. ASME, *Journal of Engineering for Industry*, November 1975, pp 1159-1165.

7. Garcia, C. B., and Li, T. Y., "On the Number of Solutions to Polynomial Systems of Equations", *SIAM J. Numer. Anal.* , Vol 17, 1980, pp 540-546.

8. Garcia, C. B., and Zangwill, W. I., "Global Continuation Methods for Finding All Solutions to Polynomial Systems of Equations in N Variables", *Center for Math Studies in Business and Economics Report No. 7755* , University of Chicago, 1977.

9. Garcia, C. B., and Zangwill, W. I., "Finding All Solutions to Polynomial Systems and Other Systems of Equations", *Math Programming* , Vol 16, 1979, pp 159-176.

10. Garcia, C. B., and Zangwill, W. I., "An Approach to Homotopy and Degree Theory", *Math Oper. Res.* , Vol 4, 1979, pp 390-405.

11. Garcia, C. B., and Zangwill, W. I., "Determining All Solutions to Certain Systems of Nonlinear Equations", *Math Oper. Res.* , Vol 4, 1979, pp 1-14.

12. Garcia, C. B., and Zangwill, W. I., *Pathways to Solutions, Fixed Points, and Equilibria* , Prentice Hall, Englewood Cliffs, NJ, 1981.

13. Griffis, M., and Duffy, J., "A Forward Displacement Analysis of a Class of Stewart Platforms", *Journal of Robotic Systems* , Vol. 6, No. 6, Dec. 1989, pp 703-720.

14. Li, T. Y., Sauer, T., and Yorke, J. A., "Numerical Solution of a Class of Deficient Polynomial Systems", *SIAM J. Numer. Anal.* , Vol 24, 1987, pp 435-451.

15. Morgan, A. P., "A Method for Computing All Solutions to Systems of Polynomial Equations", *ACM Trans. Math Software* , Vol 9, 1983, pp 1-17.

16. Morgan, A. P., "A Transformation to Avoid Solutions at Infinity for Polynomial Systems", *Appl. Math. Comput.* , Vol 18, 1986, pp 77-86.

17. Morgan, A. P., "A Homotopy for Solving Polynomial Systems", *Appl. Math. Comput.* , Vol 18, 1986, pp 87-92.

18. Morgan, A. P., *Solving Polynomial Systems Using Continuation for Scientific and Engineering Problems* , Prentice Hall, Englewood Cliffs, NJ, 1987.

19. Morgan, A. P., and Sommese, A. J., "A Homotopy for Solving General Polynomial Systems that Respects m-Homogeneous Structures", *Appl. Math. Comput.* , Vol 24, 1987, pp 101-113.

20. Morgan, A. P., Sommese, A. J., Watson, L. T., "Finding All Solutions to Polynomial Systems Using HOMPACK", *GMR-6109* , General Motors Research Laboratories Publication, Warren, MI, January 1988.

21. Morgan, A. P., and Wampler, C. W., "Solving a Planar Four-Bar Design Using Continuation", *Proceedings of the 15th ASME Design Automation Conference* , Vol. 2, 1989, pp 409-416.

22. Subbian, T., and Flugrad, D., R., "Four-Bar Path Generation Synthesis By a Continuation Method", *Proceedings of the 15th ASME Design Automation Conference* , Vol. 2, 1989, pp 425-432.

23. Tsai, L. -W, and Lu, J. -J., "Coupler-Point-Curve Synthesis Using Homotopy Methods", *Proceedings of the 15th ASME Design Automation Conference* , Vol. 2, 1989, pp 417-423.

24. Tsai, L. -W.,and Morgan, A. P., "Solving the Kinematics of the Most General Six- and Five-Degree-of-Freedom Manipulators by Continuation Methods", *Journal of Mechanisms, Transmissions, and Automation in Design* , Vol. 107, June 1985, pp 189-200.

25. Wampler, C. W., Morgan, A. P., and Sommese, A. J., "Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics", *GMR-6109* , General Motors Research Laboratories Publication, Warren, MI, August 1988.

26. Watson, L. T., "A Globally Convergent Algorithm for Computing Fixed Points of C^2 Maps", *Appl. Math Comput.*, , Vol. 5, 1979, pp 297-311.

27. Watson, L. T., "Numerical Linear Algebra Aspects of Globally Convergent Homotopy Methods", *SIAM Review* , Vol. 28, No. 4, December 1986, pp 529-545.

28. Watson, L. T., Billups, S. C., Morgan, A. P., "HOMPACK: A Suite of Codes for Globally Convergent Homotopy Algorithms", *ACM Trans. Math Software* , Vol. 13, No. 3, Sept. 1987, pp 281-310.