Enumeration and Analysis of Variable Geometry Truss Manipulators

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ABSTRACT

The variable-geometry truss, or VGT, is a generalization of Stewart's most basic platform manipulator concept. In this paper, all possible basic forms of the truss-type manipulator are enumerated in this paper. These basic forms, or unit cells, are shown to be the building blocks from which all truss-type manipulators are formed. Several manipulator concepts based on the unit cell construction are then presented as examples, including a new and potentially superior form of Stewart's Platform. Finally, a general method for the forward kinematic analysis of VGT unit cells is described.

INTRODUCTION

Many types of parallel-actuated manipulators have been proposed (Hunt, 1983), but few of them have found broad application. Stewart's platform (Stewart, 1965-66) is a notable exception, having been widely adopted for use in vehicle simulators and other platform control tasks. In its simplest and most elegant form, Stewart's platform is a variable-geometry octahedral truss with two triangular platforms connected by six extensible legs. Such an arrangement yields an inherently strong manipulator, because all the members are loaded in pure tension or compression. In general, a variable-geometry truss (VGT) can be defined as a statically determinate truss that has been modified to contain some number of variable-length members. The number of these variable members is equal to the number of degrees of freedom of the device.

VGTs have been studied for their potential as as adaptive or collapsing space structures. The devices proposed for these applications are typically symmetric, constructed of repeating identical cells, and have exceptional stiffness to weight ratios. Most VGTs of this type can be folded down and stored very compactly, an important feature for space applications. Some of the typical space applications that have been envisioned include booms to position equipment, berthing devices, serpentine structures to position and support a transfer tunnel, and supports for space antennae. The use of VGTs for such space and military applications have been discussed in detail by Cox and Nelson (1982), Rockwell International (1982), Miura, Furuya and Suzuki (1984), Miura and Furuya (1985), and Rhodes and Mikulas (1985).

Another interesting application of VGTs is as a manipulator arm or robot. The geometry that has been considered most suitable for this purpose is the octahedral truss (Miura and Furuya, 1985), (Rhodes and Mikulas 1985), (Reinholtz and Gokhale, 1987). By changing the lengths of the extensible members, the manipulator arm can vary its configuration in three-dimensional space. The VGT manipulator arm can accomplish all the functions of current articulated manipulator arms, and it also has the advantage of having higher stiffness. The use of octahedral VGTs as joints in manipulators has also been studied (Arun and Padmanabhan, 1988). Recently, techniques have been developed to precisely control the end position of long chain VGT manipulators. These techniques do not require the specification of all intermediate link variables (Salerno, 1989).

An ideal truss is composed exclusively of two-force members ; no bending moments or torques can be transmitted at the joints. This means that the relative rotations of adjoining links must occur at a single point, either through spheric joints, or through a set of revolute joints with intersecting axes. This is a difficult requirement to satisfy exactly in practice. Many trusses are approximate in the sense that some joint offset is present and hence small bending moments are transmitted between links. All the derivations presented in this paper are based on the assumptions that the truss under consideration is ideal, and that adjoining links are connected by spheric joints.

BASIC CELL TYPES

The study of basic static truss and VGT geometries is not new. Some of these geometries were studied by the greeks, notably Euclid, over two thousand years ago. More recent and elaborate developments in the study of such geometries are about a century old. Some of these developments are described by Coxeter (1962).

All VGTs can be classified as *polyhedra*, i.e., geometrical figures bounded by planes. When none of the bounding planes penetrate the interior, the polyhedron is *convex*. A convex polyhedron is said to be *regular* if its faces are regular and equal and its solid angles are equal. If the faces of a polyhedron are p-gons, q surrounding each vertex, the polyhedron is denoted by $\{p,q\}$. For example, the tetrahedron is denoted by $\{3,3\}$. There are only five regular, convex polyhedra - the platonic solids. They are the tetrahedron $\{3,3\}$, the octahedron $\{3,4\}$, the cube $\{4,3\}$, the icosahedron $\{3,5\}$, and the dodecahedron $\{5,3\}$.

All VGTs are composed of some combination of fundamental units, hereafter referred to as unit cells. The basic VGT unit cell types are all *deltahedra*. A *deltahedron* is any polyhedron whose faces are triangles. In addition to the tetrahedron, the octahedron and the icosahedron there are many other convex deltahedra composed of triangles. Williams (1970) has depicted cells having 6, 10, 12, 14 and 16 faces.

The necessary and sufficient rules for determining admissible VGT unit cells are enumerated below. The first rule deals with the mobility, or number of degrees of freedom, of the device. A truss must have a gross mobility less than or equal to zero (DOF ≤ 0) to be a structure. Mobility equal to zero indicates a statically-determinate structure, while mobility less than zero indicates a statically-indeterminate structure. A statically-indeterminate truss possesses more constraints than necessary to maintain the structure. Hence, it is not possible to independently actuate the links of such a device. VGTs must therefore be formed from statically-determinate trusses.

The mobility of a spatial mechanism such as a VGT is given by

$$\text{Mobility} = 6(N-1) - 5f_1 - 4f_2 - 3f_3 - 2f_4 - f_5, \qquad [1]$$

where

N = number of links,

f_n = number of joints having n degrees of freedom.

The second rule applied is Euler's Law, a law all polyhedra must satisfy. In three-dimensional space, Euler's Law may be expressed as (Coxeter,1962):

$$N - L + T = 2 \qquad [2]$$

where

N = number of vertices,

L = number of edges,

T = number of faces.

As noted earlier, for an ideal truss to be formed, no bending moments or torques can be transmitted from one link to the next. Furthermore, links must be two-force members when the truss is subjected to static loads. With these requirements, only binary links connected by spheric joints or their equivalents are possible in a truss. It should also be noted that the number of links, vertices, and spheric joints present in a truss must be related because every link must terminate at a joint at either end (2 N joints), but k links meeting at a vertex form only $k-1$ joints. This gives

$$S = 2L - N \qquad [3]$$

where S is the number of spheric joints.

The mobility equation [1] can be simplified as follows for structures having only N binary links and S spheric joints.

$$M = 5L - 3S - 6 \qquad [4]$$

Note that N freedoms have been subtracted to account for the idle rotation of each S-S binary link about its own axis. Substituting S from eqn. [3] into eqn. [4], setting $M = 0$, and solving for N gives

$$N = \frac{L+6}{3} \qquad [5]$$

Only integer combinations of L and N that satisfy eqn. [5] are admissible. All possible combinations of links, joints and triangles from L = 3 to L = 18 are enumerated in Table 1. Clearly, unit cells can be formed for any number of vertices beyond five, with the number of links increasing by three for each added vertex. The number of faces, T, in the table is calculated from Euler's Law, eqn. [2]. It should be noted that viable combinations beyond the tetrahedron can always be formed as dipyramids. For example, the octahedron can be formed as two four-sided pyramids with a common quadrilateral base. This has led to the misnomer of pyramid-pyramid to describe the octahedral truss. These pyramids cannot be separated without destroying the basic unit, hence the pyramid is not a basic unit. This is an important distinction, because basic units are the key to the design and analysis of VGTs. It should also be noted that viable unit cells made up of nine or more links can always be formed as combinations of simpler units in addition to the basic unit cells. For example, $L = 12$, $N = 6$, $T = 8$ can produce an octahedron (a unit cell), or three interconnected tetrahedra (not a valid unit cell).

As an example, consider the pentagonal dipyramid (decahedral unit cell) shown in Fig. 1. The decahedron has 15 links and 23 spheric joints (L = 15 , S = 23). Hence, using the mobility equation, eqn. [4] , gives

$$\text{Mobility} = 5(15) - 3(23) - 6 = 0.$$

L	N	T	S	Comment
3	3	1	3	**Triangle**
4	10/3	-	-	Not Possible
5	11/3	-	-	Not Possible
6	4	4	8	**Tetrahedron**
7	13/3	-	-	Not Possible
8	14/3	-	-	Not Possible
9	5	6	13	Two Tetrahedra
10	16/3	-	-	Not Possible
11	17/3	-	-	Not Possible
12	6	8	18	**Octahedron**
13	19/3	-	-	Not Possible
14	20/3	-	-	Not Possible
15	7	10	23	**Decahedron**
16	22/3	-	-	Not Possible
17	23/3	-	-	Not Possible
18	8	12	28	**Dodecahedron**

where

N = number of vertices,

L = number of edges/links,

T = number of faces,

S = number of spheric joints.

TABLE I

Hence, the decahedron has a gross mobility of zero and is statically determinate. In addition to satisfying the mobility and Euler equations, basic VGT unit cells must not be composed of combinations of simpler unit cells. Thus the triangular dipyramid (table entry with $L = 9$ and $N = 5$) cannot be considered a basic cell because it can always be separated into two tetrahedra. Therefore, the octahedron is the simplest unit cell beyond the tetrahedron. Some of the basic cells are shown in Fig. 1.

CONSTRUCTING VGTS FROM UNIT CELLS

It has been shown that only a few basic and distinct truss unit cells exist, and that every statically-determinate truss, no matter how complex, must be formed from combinations of these cells. Furthermore, any link of a statically-determinate truss can be made extensible, and each such extensible link will add one degree-of-freedom to the device. Thus a tetrahedron can have upto six extensible links and six degrees of freedom, an octahedron can have twelve, and so on. Note that an extensible link would, in the classic sense, be considered two coaxial links connected through a prismatic joint. In the context of a VGT, it is more convenient to think of a single extensible link, although the two views are ultimately equivalent.

Many practical examples of planar VGTs can be found. These are based on triangular frames, typically having one extensible link, such as the device shown in Fig. 2. This is the basic mechanism used in equipment ranging from backhoes to manipulators. In all such devices, the extension of link 3 (l_3) is used to control the angle θ. Clearly l_2 could also be made extensible and the combined motion of l_2 and l_3 could be used to control the position of point P. It is a simple matter to extend this logic to produce a planar variable-geometry-truss manipulator such as the one depicted in Fig. 3.

Spatial VGT manipulators must be composed of some combination of the three-dimensional unit cells. Even when considering only the tetrahedron, octahedron,

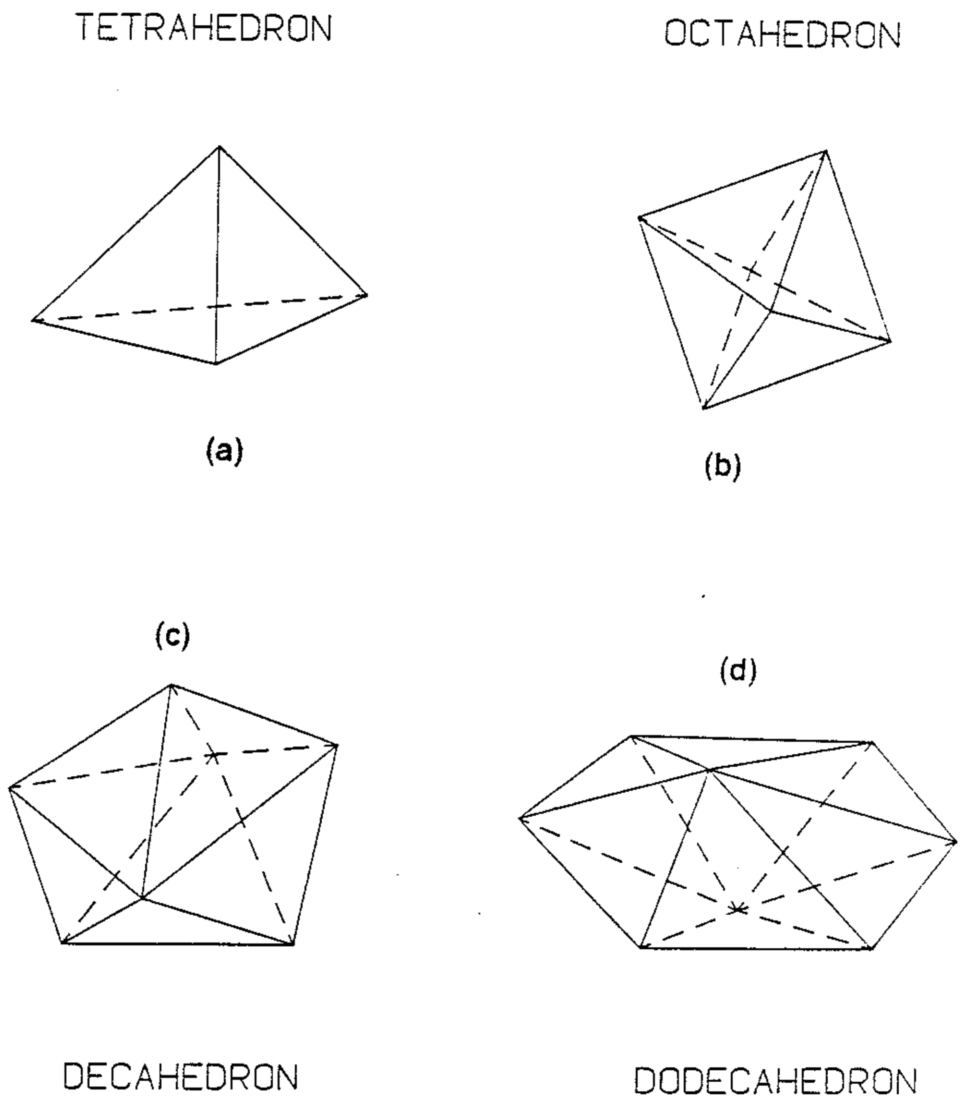


Figure 1. Basic VGT Unit Cells

decahedron and dodecahedron, the possible useful combinations of cells and actuation schemes become complicated. The single cell tetrahedron, by itself, has limited usefulness as a manipulator. The three side members can be actuated to control the position of the apex, P, as shown in Fig. 4.

In a similar manner, the six side members of the octahedron can be actuated to yield a well known form of Stewart's Platform (Stewart, 1965-66), as shown in Fig. 5. It should be noted from mobility results that the same combinations of links, joints, and faces that produce the octahedron ($L = 12$, $N = 6$, $T = 8$) can also produce three interconnected tetrahedra. The three tetrahedral configuration leads to a distinctly different form of the platform manipulator, as shown in Fig. 6.

Like the conventional Stewart's platform, actuation of the newly-proposed platform manipulator is through the six extensible legs connecting the platform to the ground. However, the structure is now kinematically much simpler than the octahedron, which has an eighth degree polynomial as its governing equation (Griffis, Duffy, 1989). Instead, the new platform manipulator requires only the solution of three independent quadratic equations that govern the three tetrahedra. Furthermore, the tetrahedron-based platform may in some cases be superior to the octahedron in load-carrying capacity and stiffness. Although Stewart's platform and the newly proposed platform are similar in appearance, they are fundamentally different structures.

Another embodiment of the octahedron that results in a potentially useful manipulator is shown in Fig. 7. This has been referred to by many names, the most appropriate of which seems to be the "double octahedral" VGT. In this arrangement, two octahedra are joined at a common triangular face, and the three links of this common face are actuated.

When the two octahedra are made symmetric about the actuated plane, the VGT can be made to collapse nearly flat by sufficiently extending the three actuated links. By repeating this basic structure, a very long, collapsible, snake-like manipulator can be built. By nature of the basic truss structure inherent in these devices, it is conceivable that a serpentine arm having perhaps fifty degrees of freedom could be designed and constructed.

Many other interesting combinations of the basic unit cells are possible. Long chain devices can be readily formed from tetrahedra, octahedra, decahedra and dodecahedra. In fact, conventional static trusses using each of these units as repeating substructures are commonplace, although they may not be readily identifiable. For example, the so-called cube truss common to many structures can be composed of dodecahedra, as shown in Fig. 8. The cube truss can also be formed from various combinations of tetrahedra and octahedra.

FORMULATION OF KINEMATIC EQUATIONS

The kinematic constraints that exist any VGT can always be expressed as a system of quadratic equations having the following form :

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = L_{ij}^2, \qquad [6]$$

where

(x_i, y_i, z_i) = coordinates of node i,

(x_j, y_j, z_j) = coordinates of node j,

L_{ij} = length of the link connecting nodes i and j.

This equation simply constrains node j to lie a fixed distance L_{ij} from node i. In general, this equation will be written once for each moving link of a truss cell. It will be

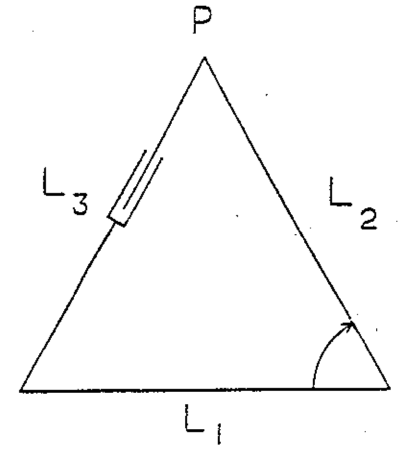


Figure 2 - Planar VGT

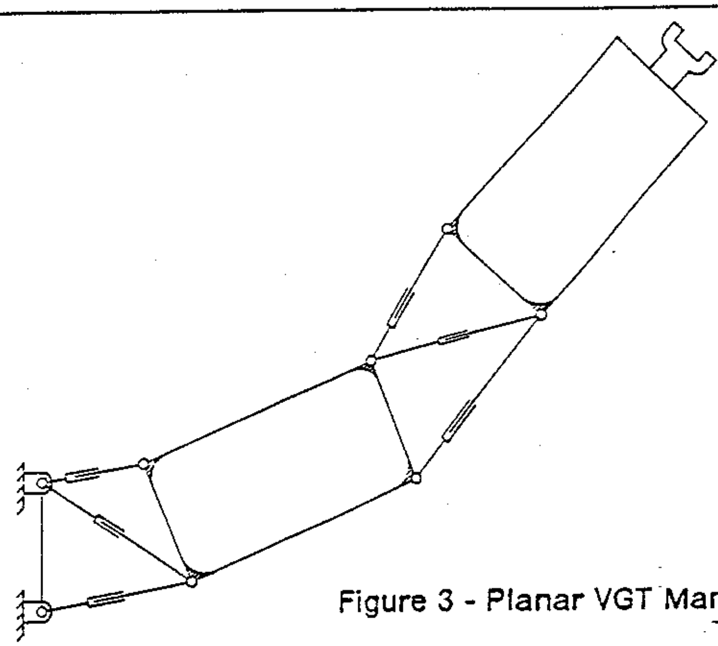
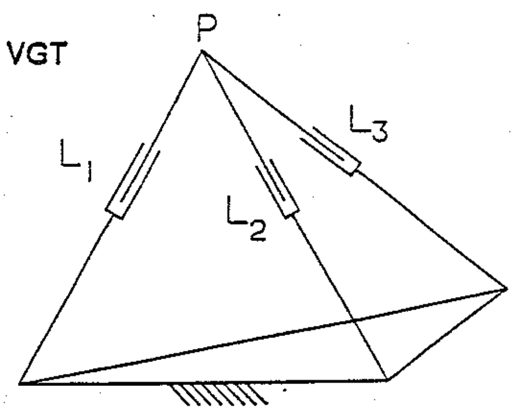


Figure 3 - Planar VGT Manipulator

Figure 4 - Tetrahedral VGT



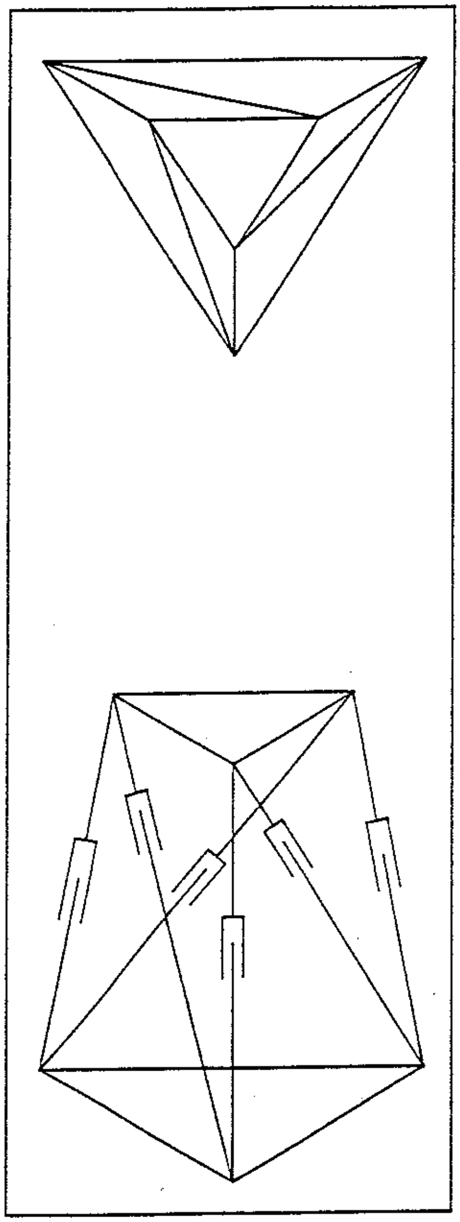


Figure 5 - Stewart's Platform

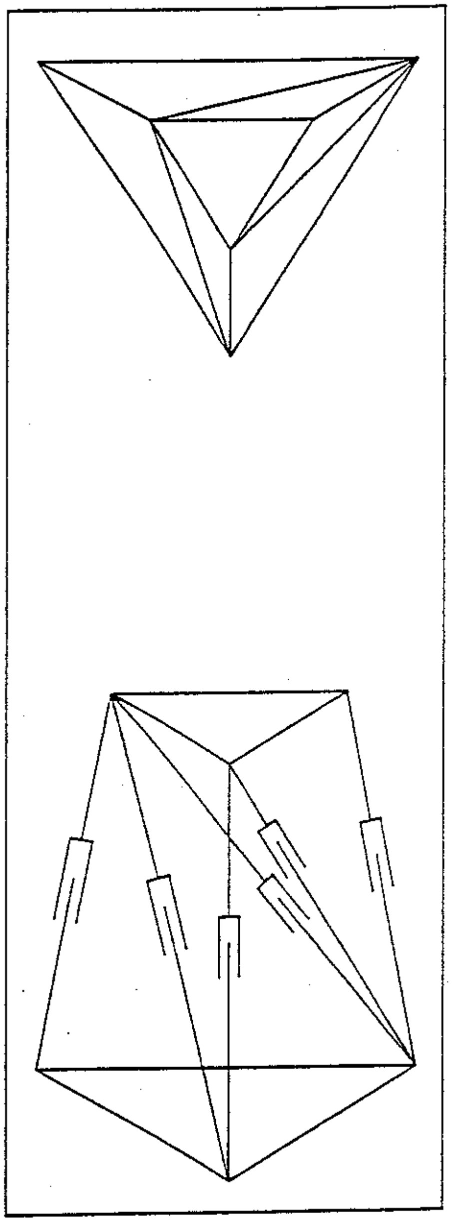


Figure 6 - Tetrahedon-based Platform

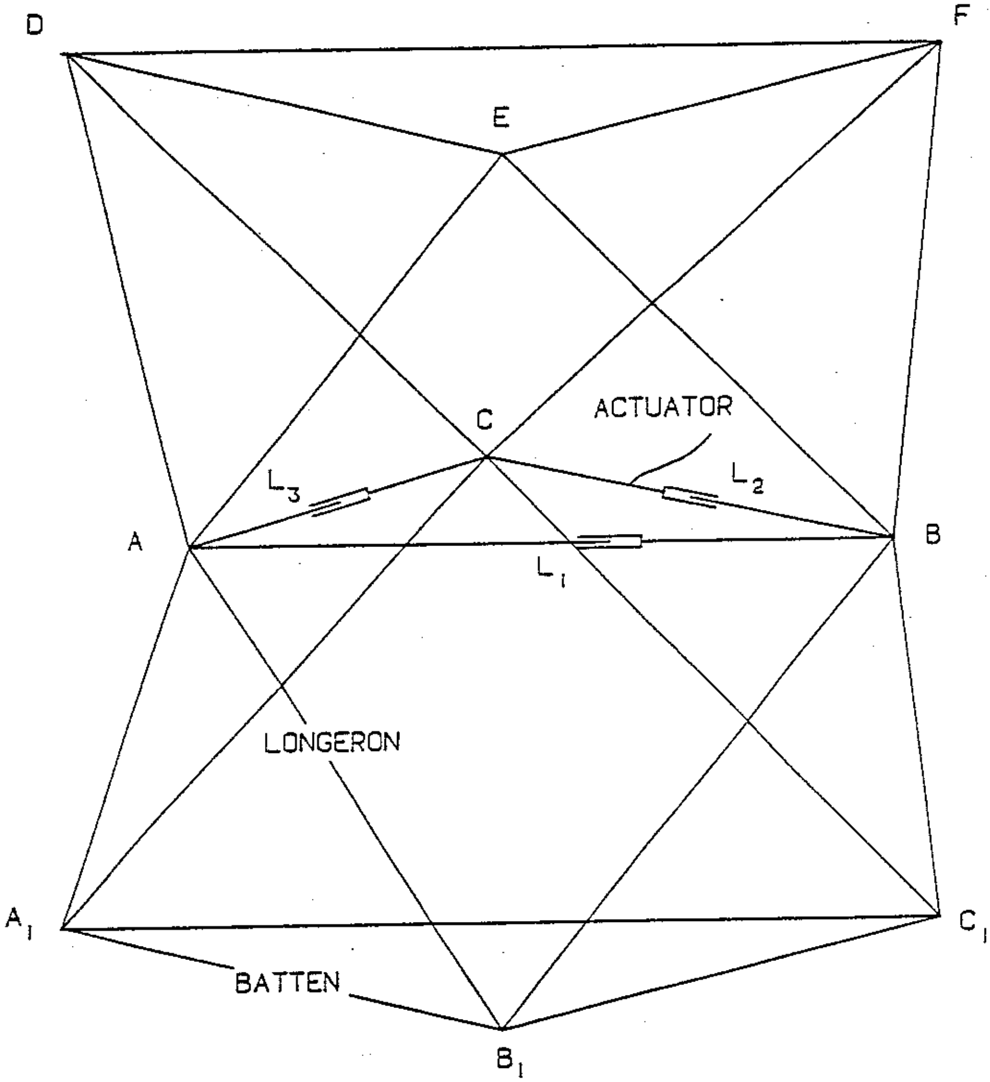


Figure 7 - Double Octahedral VGT

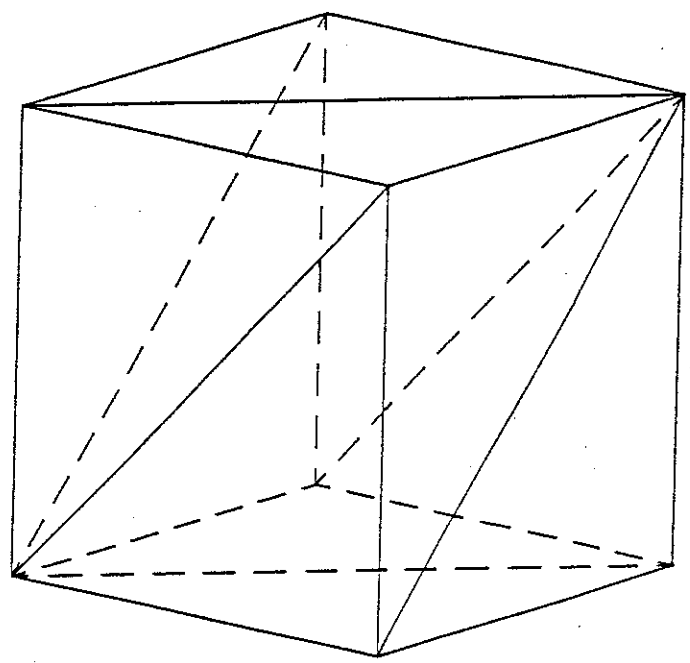


Figure 8 - Dodecahedron-based Cube Truss

assumed that one triangular face of the cell forms the reference plane, and so the constraint equation will not be written for the node pairs in this plane. For example, the tetrahedron has three moving links if one triangle is taken as the reference. Therefore, three quadratic equations can be written containing the unknown coordinates at point P (refer to Fig. 4). Using this reasoning, the number of quadratic equations to be solved simultaneously for the common unit cells can be quickly determined, and are shown in table 2.

N	Unit Cell	No. of Quadratic Eqns.
6	Tetrahedron	3
12	Octahedron	9
15	Decahedron	9
18	Dodecahedron	15

TABLE II

While this approach is effective because of the simplicity of forming the equations, numerical solution of the resulting systems of equations is often difficult, or impossible by conventional methods. This has led to the consideration of both alternate formulations, and new numerical solution methods, as described below.

The tetrahedron can also be modeled as a three link RSS (Revolute-Spheric-Spheric) chain as shown in Fig. 9b. Given the length L and the height h, and the coordinates of the base triangle vertices, the forward problem reduces to the determination of the unknown angle θ. A constraint equation can be written for that purpose:

$$(\underline{d} - \underline{b}_0) \cdot (\underline{d} - \underline{b}_0) = L_3^2, \qquad [7]$$

where $\underline{d} = \underline{d}_0 + [R_{u,\theta}](\underline{d}_1 - \underline{d}_0)$, and $[R_{u,\theta}]$ is a rotation matrix describing a rotation θ about an axis u.

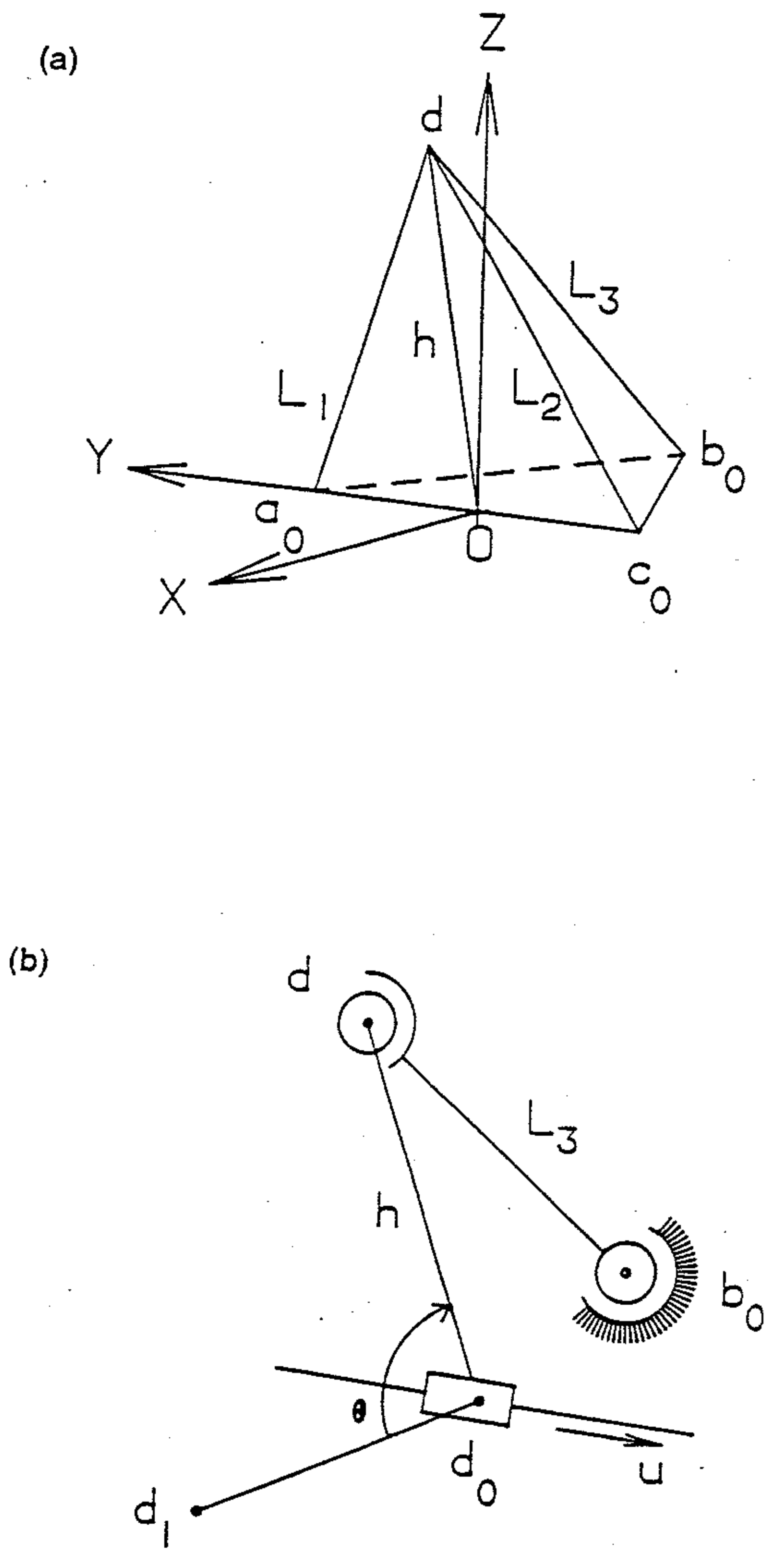


Figure 9 - Tetrahedral Unit Cell & Associated Kinematic Model

This constraint equation represents the intersection of a circle (about the revolute joint) and a sphere. The above equation reduces to a quadratic $At^2 + Bt + C = 0$, where A, B, and C are functions of the link lengths and $t = \tan\frac{\theta}{2}$. This equation can easily be solved using the quadratic formula.

The tetrahedron is the only VGT unit cell whose forward kinematics can be solved in closed form.[1] When higher order VGTs like the decahedron and the dodecahedron and others with more faces are considered, the equations describing the forward kinematics become increasingly complex; they are all nonlinear and are of high degree. It is impractical and in many cases impossible to attempt to eliminate variables and obtain a single equation in one unknown. Also, the method used for elimination is important, as more than one elimination can produce a resultant which contains the required equation multiplied by an extraneous factor that is very difficult to find (Duffy and Rooney, 1975). Once this happens, it becomes unclear as to which roots lead to the closure of the real mechanism. It is far more practical to solve the systems of equations numerically. Commonly, such systems are solved by an iterative numerical method, usually the Newton-Raphson method. The shortcoming of the n-dimensional Newton-Raphson method is that excellent initial guesses are required to ensure convergence. Success is never guaranteed because there is no sure way of making a good initial guess. The method also runs into problems in finding and distinguishing multiple solutions.

On the other hand, recently developed methods in homotopy continuation for polynomials (polynomial continuation) are not only global, but also exhaustive; i. e., they do not require good initial guesses and also guarantee convergence to all sol-

[1] It may be possible to obtain the governing polynomial equation of more complex cells in closed form, but the roots of that equation must be obtained numerically.

utions. Homotopies are a traditional part of topology and only recently have begun to be used for practical numerical computation. The homotopy continuation method and the mathematical theory behind it is described in (Wampler, Morgan, Sommese, 1988). A summary of the method and an explanation of its use to solve the forward kinematics of complex VGTs is given in a companion paper (Arun, Reinholtz, Watson, 1990).

CONCLUSIONS

This paper has presented the fundamental theory of variable-geometry trusses, including criteria to identify basic unit cells. Guidelines for building different VGTs using these basic units are also presented. One of the concepts described is a new platform-type manipulator similar to the Stewart's platform. A general method for the forward kinematic analysis of VGTs is described, and the technique of homotopy continuation is introduced as a means for solving the forward kinematics of VGT unit cells.

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