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Abstract. The necessary and sufficient conditions of local and global optimization are derived for constrained resource allocation with continuous variables and objective functions of the forms $\max_i\{f_i(x_i)\}$ and $\min_i\{f_i(x_i)\}$ where $\{f_i\}$ can be nondifferentiable, nonmonotone, nonconvex, multimodal functions. All previous theoretical results, which are sufficient conditions for global optimization with monotone $\{f_i\}$, are special, restrictive cases of the new criteria. The powerful criteria enable complete determination of *all* the global optimal solutions, thus allowing further lexicographic optimization. The criteria also enable determination of all the local maxima and minima, a previously unaddressed facet of the solution, thus providing illuminating information on the behavior of the objective function and its overall "topography," which could be useful in suboptimal multiple-criteria trade-offs. The new results admit a straightforward graphical interpretation and implementation, which facilitates their utilization and extends their applicability to practical problems where $\{f_i\}$ are specified only in graphical formats derived from empirical or simulation data. Except for the mild and practically insignificant restrictions of continuity and "local monomodality" retained on $\{f_i\}$ by the analysis, the results of this paper constitute the complete and definitive solution of the problem.

Categories and Subject Descriptors: G.1.6 [**Mathematics of Computing**]: Optimization--*constrained optimization, nonlinear programming*

General Terms: Theory, Algorithms, Performance

Additional Key Words and Phrases: Resource allocation, minimax optimization, maximin objective functions, local optimization, relative extrema, necessary and sufficient conditions, nonconvex analysis, graphical techniques, lexicographic optimization, suboptimal trade-offs, multicriterion optimization, parallel processing performance

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1. INTRODUCTION

The classical resource allocation problem is an optimization problem with a single equality constraint: Given a fixed total amount of a resource, one seeks to determine its partitioning and allocation to a given number of recipient activities in such a way that an appropriately defined objective function is optimized. It is a special case of the nonlinear programming problem and is encountered in various application areas such as load distribution, computer scheduling, production planning, among many other settings. It also arises as a subproblem of more complex problems. Research on the solution of the basic problem and its variants, which has been going on in various application fields over the last three decades, has produced a number of interesting theoretical criteria as well as numerous computational algorithms [7].

The mathematical formulation of the resource allocation problem may be stated as follows:

$$\text{Optimize} \quad g(x_1,x_2,\ldots,x_n) \tag{1}$$

$$\text{Subject to} \quad \Sigma_1^n x_i = L \tag{2}$$

$$\alpha_i \leq x_i \leq \beta_i \tag{3}$$

where x_i is the apportionment of the total resource L allocated to the i-th activity, g is a real-valued objective function of the allocation vector $x=(x_1,x_2,\ldots,x_n)$ to be maximized or minimized, and α_i and β_i are specified lower and upper bounds on x_i imposed by conditions or requirements dictated by the i-th activity. If the nature of the resource is such that it is divisible down to magnitudes that are smaller than the accepted resolution level of its measurement, the resource quantity x_i is mathematically modelled as a continuous variable over the specified interval $[\alpha_i,\beta_i]$. If these values are integers, one has an integer programming problem. The problem is further categorized into classes according to the nature or type of the objective function g(x), for which specific solutions in the form of theoretical criteria or computational algorithms are developed. Typical forms of the objective function have been extensively investigated in the literature because of their occurrence in certain important classes of applications. One such form is the "separable" objective function $g(x) = \Sigma_1^n f_i(x_i)$, where each f_i is a function of the single variable x_i. Another important class is the problem whose objective function has one of the two forms:

$$g(x) = \max_i f_i(x_i) \equiv F(x) \quad \text{or} \quad g(x) = \min_i f_i(x_i) \equiv f(x) \tag{4}$$

which traditionally has been known as the "minimax" or "maximin" problem respectively, because the minimization of the first form and maximization of the second are the more common occurrences of optimization of the objective functions encountered in practice. Maximization of F(x) or minimization of f(x) is, in most cases, much easier to solve, and in some cases is trivial.

The mathematical formulation for the minimax/maximin resource allocation problem (REMAXMIN) becomes

REMAXMIN: Optimize $\quad F(x) \equiv \max_i f_i(x_i) \quad , \text{ or } \quad f(x) \equiv \min_i f_i(x_i)$ (5)

Subject to $\quad \Sigma_1^n x_i = L$ (6)

$$\alpha_i^1 \leq x_i \leq \beta_i \, . \tag{7}$$

One important application area in computer scheduling and performance analysis, which under certain conditions can be modelled by these relationships, is the optimization of the execution-time of a given workload partitioned and allocated to run on a parallel or distributed computer system with multiple processing elements [5]. In this case L represents the total workload, x_i the subload or task assigned to processor P_i, $f_i(x_i)$ the execution time of processor P_i, and F(x) the completion time of the longest-running processor which marks the total job completion time. The specified upper bound β_i may represent a limitation on the load handling capacity of processor

P_i, such as its maximum main memory allocation. The lower bound α_i may reflect a deliberate scheduling policy of assigning at least a load α_i to processor P_i if it is to be allowed to participate in the total job execution.

This paper addresses the solution of REMAXMIN stated in (5), (6), and (7). This problem has been investigated in the literature, and there are analytical criteria as well as numerical algorithms for its solution [7]. All of the available results, however, suffer from two principal drawbacks [2], [3], [7]:

1. Monotonicity of $f_i(x_i)$: The functions $f_i(x_i)$ are restricted to be either all monotone nondecreasing or all monotone nonincreasing.
2. Sufficiency but not necessity: The results present sufficient, not necessary, conditions for optimality.

These "deficiencies" may, in certain situations, seriously hamper the applicability and limit the usefulness of the available results. The criteria fail to provide any information when any of the functions $f_i(x_i)$ is nonmonotone or when its monotonicity is opposite to that of any other function $f_j(x_j)$. A case in point of such situations is to be found in the multiprocessing/distributed load allocation application mentioned above: When the individual tasks x_i assigned to different processors communicate among themselves by exchanging messages or accessing shared memory, the execution-time function $f_i(x_i)$ of processor P_i may exhibit increasing and decreasing behavior over different intervals of its domain x_i []. The fact that the available results are only sufficient, not necessary, conditions means they embody incomplete information about optimality: If no point x satisfying the sufficient conditions could be found. the results would fail to solve the optimization problem at hand; and if an optimal point could be found, there could be other optimal points that do not satisfy the conditions which therefore would remain undiscovered.

The results of this paper overcome both of the aforementioned deficiencies in the existing criteria. The new theorems presented here do not stipulate any restrictions of differentiability, monotonicity, convexity, or unimodality on the functions $f_i(x_i)$. We only require $f_i(x_i)$ to be continuous over $[\alpha_i, \beta_i]$ and to have the mild property of "local monomodality" (to be defined later) which always exists in real-world situations. Thus $f_i(x_i)$ may exhibit any number of local maxima and minima over its domain of definition. Figure 1 shows an example of a function $f_i(x_i)$ admissible under the analysis and for which the results of this paper may be applied. Furthermore, the results we derive are necessary as well as sufficient conditions of optimality. As such, they provide complete information which may be used reliably to determine all the optimal solutions of REMAXMIN, i.e., the conditions are *equivalent* to the concept of optimality.

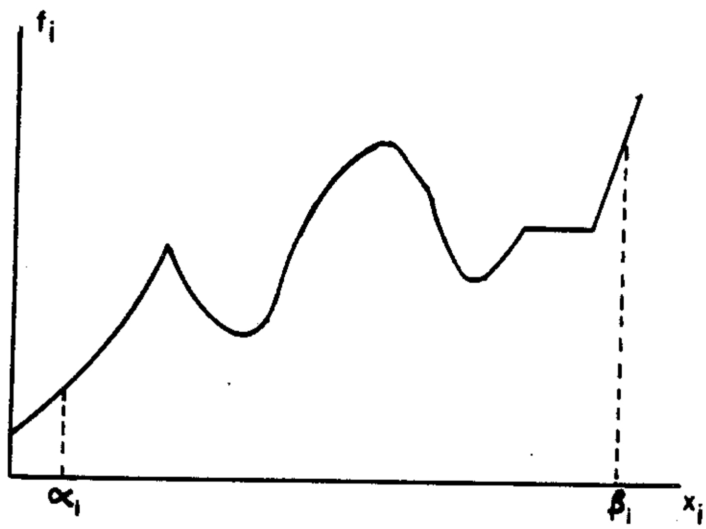


FIG. 1. Illustrating a nondifferentiable, nonmonotone, nonconvex, multimodal function $f_i(x_i)$ admissible in the analysis.

The approach we shall take in solving REMAXMIN, as stated in (5)-(7), is first to find the set of all *local* minimum points (lminp) and all *local* maximum points (lmaxp) of the objective function F(x) or f(x):

$$X_m \equiv \{x: x \text{ is a lminp of } F(x)\}$$

$$X_m \equiv \{x: x \text{ is a lmaxp of } F(x)\} .$$

Then we find the set of all global maximum points (gmaxp) and all global minimum points (gminp)

$$X_g \equiv \{x{:}x \text{ is a gminp of } F(x)\} = \{x: F(x) = \min F(x) = \min_{x \in Xm} F(x)\}$$

$$X_g \equiv \{x{:}x \text{ is a gmaxp of } F(x)\} = \{x: F(x) = \max F(x) = \max_{x \in Xm} F(x)\} .$$

The last equalities indicate how knowledge of the sets X_m and x_m of local minima and maxima can be used to find the sets X_g and X_g of global minima and maxima respectively.

It should be emphasized here that our interest in determining all the local minima and maxima is not necessarily only for the purpose of determining the global extrema of the objective function F(x). Knowledge of all local extremum points of F(x) constitutes an important problem in its own right which may have significant relevance to the underlying practical problem modelled by REMAXMIN. Recall that the objective function of any optimization problem is a "performance index" whose behavior and variation is the focus of attention of the practical problem under consideration. Finding the global extremum values is only one aspect of determining the behavior of F(x), while finding all local as well as global extrema is tantamount to charting a comprehensive picture of the "topography" of F(x) and illuminating its behavior over the entire domain of the feasible set. One practical situation where knowledge of all local extrema might prove to be quite useful is when the selection of a resource allocation solution x is guided not only by the value it imparts to the objective function

F(x) but also by some additional or secondary criterion of desirability. In such cases one may opt to choose a local optimal solution that is suboptimal with respect to the primary objective function but is optimal with respect to the secondary objective function. To illustrate, consider again the parallel processing application with $n=4, L=8$, and suppose we find the set X_m to be comprised of the following three local minima of the execution-time function F(x):

$$F(8,0,0,0) = 100 \quad , \quad F(4,2,1,1) = 110 \quad , \quad F(2,2,2,2) = 101 \; .$$

Evidently, the allocation $x=(8,0,0,0)$ is the global minimum solution. But we might choose instead the local minimum $x=(2,2,2,2)$ as the preferred "optimal" solution, trading off a one percent increase in execution-time for a far better balancing of load allocation among the four processors, assuming load balancing is our secondary objective function.

To recapitulate, there are three significant new features offered by the approach and results of this paper in comparison with previous investigations of REMAXMIN:

1. Relaxation of the monotonicity restriction on $f_i(x_i)$, allowing the functions to be nonmonotone, nonconvex, nondifferentiable, and multimodal.
2. Determining the conditions for local as well as global minima and maxima, thus charting a comprehensive picture of the objective function behavior.
3. Providing complete information about optimality in the form of necessary as well as sufficient conditions for optimality.

The analytical significance of these new features may be further appreciated if one looks beyond the realm of the REMAXMIN problem at hand into the wider discipline of nonlinear mathematical programming, where the vast majority of theoretical and algorithmic results for general classes of problems are found to deal only with global extrema, are either necessary or sufficient conditions, and almost invariably impose some form of convexity/concavity restriction on the objective function [1].

2. DEFINITIONS AND PRELIMINARY ANALYSIS

2.1 Problem Formulation

Given the set of n functions $\{f_i(x_i)\}$ where each $f_i(x_i)$ is a real-valued function of the real variable x_i, defined and continuous at every point in the nonnegative interval $[a_i,b_i]$. Consider the real-valued *objective function* F(x) of the n variables $x=(x_1,x_2,\ldots,x_n)$ defined as $F(x) \equiv \max_i f_i(x_i)$. The domain of F(x) is restricted to the *constraint set* $C(L,\alpha,\beta)$ representing the collection of n-dimensional points x satisfying the following constraints

$$C(L,\alpha,\beta) \equiv \{x: \Sigma_1^n x_i = L \quad , \quad x_i \in [\alpha_i,\beta_i]\} \tag{8}$$

where $\alpha=\{\alpha_i\}$ and $\beta=\{\beta_i\}$. We are interested in studying the behavior of the objective function F(x) over the domain C and determining all its local and global minimum and maximum points in C. We shall require the following relationship to hold among L, $\{\alpha_i\}$, and $\{\beta_i\}$:

$$\Sigma_1^n \alpha_i < L < \Sigma_1^n \beta_i \tag{9}$$

for otherwise C as defined in (8) would be empty if $L < \Sigma\alpha_i$ or if $L > \Sigma\beta_i$, and C would degenerate into a single point if $L=\Sigma\alpha_i$ or if $L=\Sigma\beta_i$, in which case the problem becomes trivial. The constraint set $C(L,\alpha,\beta)$ represents a subset of the hyperplane $x_1+x_2+\ldots+x_n=L$ in n-dimensional space and is also referred to as the *feasible set.*

2.2 Attainable Feasible-Set Bounds

The parameters α_i and β_i are *a priori* specified lower and upper bounds on the variables x_i, which may be dictated by practical considerations or physical limitations of the resource allocation problem. If, for instance, x_i represents the portion of the total load L allocated to processor P_i in a multiprocessor system, then β_i may be set by the need to restrict the loading of P_i from getting too close to the saturation point of some of its local resources such as main memory, while α_i may reflect a deliberate policy of not engaging processor P_i unless its loading is above a certain minimum level justifying its participation in job execution. Define

$$a_i \equiv \min_{x\in C(L,\alpha,\beta)} x_i \quad , \quad b_i \equiv \max_{x\in C(L,\alpha,\beta)} x_i \; . \tag{10}$$

Note that C is a compact set since it is the intersection of the compact n-cell $\{x: x_i\in[\alpha_i,\beta_i]\}$ and the closed hyperplane $\{x: \Sigma x_i=L\}$ [9]. Therefore the extremum values a_i and b_i of the continuous function x_i are *attainable* for some values of $x\in C$ [9]. Thus

$$a_i = x_i^1 \quad , \quad b_i = x_i^2 \quad , \quad \text{some } x^1,x^2 \in C \; .$$

Since $x_i^1\in[a_i,b_i]$ and $x_i^2\in[a_i,b_i]$, (10) implies that

$$a_i \geq \alpha_i \quad , \quad b_i \leq \beta_i \quad , \quad [a_i,b_i] \subseteq [\alpha_i,\beta_i] \; . \tag{11}$$

Letting $a\equiv(a_1,a_2,...,a_n)$ and $b\equiv(b_1,b_2,...,b_n)$, we now show that the set $C(L,\alpha,\beta)$ remains unchanged if α,β are replaced by a,b respectively, i.e.,

$$C(L,a,b) = C(L,\alpha,\beta) \; . \tag{12}$$

If $x\in C(L,a,b)$, then $x_i\in[a_i,b_i]\subseteq[\alpha_i,\beta_i]$, hence $x\in C(L,\alpha,\beta)$. If $x\in C(L,\alpha,\beta)$, then (10) implies $x_i\geq a_i$, $x_i\leq b_i$ and $x_i\in[a_i,b_i]$; hence $x\in C(L,a,b)$. We shall refer to $[\alpha_i,\beta_i]$ as the *specified* interval constraint on x_i and to $[a_i,b_i]$ as the *attainable* interval constraint. In the remainder of this paper we shall always assume that, for any given REMAXMIN problem, the specified interval constraints are replaced by the corresponding attainable intervals as defined in (10) with the problem remaining invariant under this replacement. Note that the problem of determining a_i and b_i, as indicated in (10), constitutes a linear programming problem with upper and lower bounds on the variables x_i, which can be solved by known techniques such as the modified simplex method [4],[8]. Recent results by Haddad [6] have provided more efficient methods for determining the attainable bounds a_i and b_i from the specified bounds α_i and β_i.

2.3 Neighborhoods and Increments

For a given $x \in C$ and $\delta > 0$, we define $N_C(x,\delta)$ as the δ-neighborhood in C(L,a,b) of point x

$$N_C(x,\delta) \equiv \{(x+\Delta x) \in C : \|\Delta x\| \leq \delta\} \tag{11}$$

where $\Delta x = (\Delta x_1, \Delta x_2, ..., \Delta x_n)$ and $\|\Delta x\|$ is the Euclidean norm of Δx:

$$\|\Delta x\| = (\Sigma_1^n (\Delta x_i)^2)^{1/2} \leq \delta . \tag{12}$$

From (12) one has

$$|\Delta x_i| \leq \delta \qquad \text{for all i, whenever } (x+\Delta x) \in N_C(x,\delta) .$$

Define $\Delta f_i(x_i, \Delta x_i)$ as the increment in f_i due to the increment Δx_i in x_i:

$$\Delta f_i(x_i,\Delta x_i) \equiv f_i(x_i+\Delta x_i) - f_i(x_i) . \tag{13}$$

For convenience, we shall let the dependence of Δf_i on x_i and Δx_i be understood, and write $\Delta f_i \equiv \Delta f_i(x_i,\Delta x_i)$. Similarly, we define $\Delta F(x,\Delta x)$ as the increment in F(x) due to the increment Δx in x

$$\Delta F(x,\Delta x) \equiv F(x+\Delta x) - F(x) . \tag{14}$$

Again, we write $\Delta F \equiv \Delta F(x,\Delta x)$.

2.4 The Subsets X and $\bar{X}$

Consider any point $(x_1,x_2,...,x_n) \in C$. For convenience we shall use the symbol x to denote the unordered set of components x_i, i.e., we let $x \equiv \{x_i\}$. We now partition the set $x = \{x_i\}$ into two subsets X and $\bar{X}$ defined as follows:

$$X \equiv \{x_j \in x : f_j(x_j) = F(x) \equiv \max_i f_i(x_i)\} \neq \phi \tag{15}$$

$$\bar{X} \equiv \{x_j \in x : f_j(x_j) < F(x)\} . \tag{16}$$

Note that while X is always a nonempty set, $\bar{X}$ may be empty and that

$$x = X + \bar{X}$$

where the "+" operator is used to denote the union operator for sets. We shall refer to X as the *peak subset* of x because $f_j(x_j)$ assumes the peak value F(x) of $\{f_i\}$ whenever $x \in X$. The subsets X and $\bar{X}$ will be used later in the statement of the main results of this paper. As an example, let $x=(x_1,x_2,x_3,x_4,x_5)$ and $\{f_i(x_i)\}=(1,4,2,3,4)$; then $X=\{x_2,x_5\}$ snf $\bar{X}=\{x_1,x_3,x_4\}$. If, as a second example, $\{f_i\}=(3,3,3,3,3)$, then X=x and $\bar{X}=\phi$. Define the sets of integers P and Q as follows

$$P \equiv \{j : x_j \in X\} \tag{17}$$

$$Q \equiv \{j : x_j \in \bar{X}\} . \tag{18}$$

Note that $P+Q=\{1,2,...,n\}$ and that

$$f_j(x_j) = F(x) \qquad \text{for all } j \in P \tag{19}$$

$$f_j(x_j) < F(x) \qquad \text{for all } j \in Q . \tag{20}$$

For the first example above, we have $P=\{2,5\}$ and $Q=\{1,3,4\}$; for the second example, $P=\{1,2,3,4,5\}$ and $Q=\{\phi\}$.

2.5 Continuity of F(x)

We now present a lemma that establishes the continuity of the objective function F(x) and states that, for sufficiently small values of Δx, the value of $F(x+\Delta x) = \max_i f_i(x_i+\Delta x_i)$ can be determined by considering only the values of $f_i(x_i+\Delta x_i)$ for $i \in P$ rather than for $i \in (P+Q)$.

LEMMA

If f_i *are continuous, then for any* $x \in C$

(i) *there exists a* $\delta_0 > 0$ *such that*

$$\Delta F(x,\Delta x) = \max_{i \in P} \Delta f_i(x_i,\Delta x_i) \qquad \textit{for all } \|\Delta x\| < \delta_0 . \tag{21}$$

(ii) F(x) *is continuous.*

PROOF

If $Q=\phi$, the result in (21) follows immediately since

$$F(x+\Delta x) \equiv \max_{i\in(P+Q)} f_i(x_i+\Delta x_i) = \max_{i\in P} f_i(x_i+\Delta x_i)$$

$$= \max_{i\in P}\{f_i(x_i)+\Delta f_i\} = F(x)+\max_{i\in P}\Delta f_i \qquad (22)$$

where the last step in (22) follows from $F(x)=f_i(x_i)$ for $i\in P$ in (19). If on the other hand $Q\neq\phi$, let $d=\min_{i\in Q}\{F(x)-f_i(x_i)\}>0$. The positiveness of d follows from (20). Evidently one has

$$F(x) - f_i(x_i) \geq d > 0 \qquad \text{for all } i \in Q\,. \qquad (23)$$

Since the functions f_i are continuous, we can choose, for any given i, a sufficiently small $\delta_i>0$ such that

$$|f_i(x_i+\Delta x_i) - f_i(x_i)| < d/2 \qquad \text{for all } |\Delta x_i| < \delta_i\,, \text{ any } i\,. \qquad (24)$$

Applying (24) specifically for $j\in P$ and $k\in Q$, one obtains, respectively,

$$-d/2 < f_j(x_j+\Delta x_j) - F(x) < d/2 \qquad \text{for all } |\Delta x_j| < \delta_j\,,\ j \in P$$

$$-d/2 < f_k(x_k+\Delta x_k) - f_k(x_k) < d/2 \qquad \text{for all } |\Delta x_k| < \delta_k\,,\ k \in Q\,. \qquad (25)$$

From the first of these relationships, we obtain

$$f_j(x_j+\Delta x_j) > F(x) - d/2 \qquad \text{for all } |\Delta x_j| < \delta_j\,,\ j \in P\,. \qquad (26)$$

Combining (25) with (23) we obtain

$$f_k(x_k+\Delta x_k) < f_k(x_k)+d/2 < F(x)-d/2 \qquad \text{for all } |\Delta x_k| < \delta_k\,,\ k\in Q\,. \qquad (27)$$

From (26) and (27) we deduce

$$f_{j\in P}(x_j+\Delta x_j) > f_{k\in Q}(x_k+\Delta x_k) \qquad \text{for all } |\Delta x_j| < \delta_j\,,\ |\Delta x_k| < \delta_k\,. \qquad (28)$$

We now can choose $\delta_0 = \min\{\delta_i\}$, and by letting $\|\Delta x\| < \delta_0$, one has $|\Delta x_i| \leq \|\Delta x\| < \delta_0 \leq \delta_i$, and

$$F(x+\Delta x) = \max_{i\in(P+Q)} f_i(x_i+\Delta x_i) = \max_{i\in P} f_i(x_i+\Delta x_i), \text{ for all } \|\Delta x\| < \delta_0\,,$$

the last statement being a direct consequence of (28). Using the fact that $f_i(x_i)=F(x)$ for $i\in P$, the required result follows

$$F(x+\Delta x) = \max_{i\in P}\{f_i(x_i)+\Delta f_i\} = F(x)+\max_{i\in P}\Delta f_i$$

$$F(x+\Delta x) - F(x) = \max_{i\in P}\Delta f_i = \Delta F \qquad \text{for all } \|\Delta x\| < \delta_0 . \tag{29}$$

The continuity of F(x) follows directly from the result just proven. Given any $\epsilon>0$, the continuity of f_i ensures that we can choose a sufficiently small δ_0 such that $|\Delta f_i|<\epsilon$ for all $i\in P$, and (29) implies $|\Delta F|<\epsilon$.

2.6 Locally Monomodal Functions

So far the only condition imposed on the functions $f_i(x_i)$ is that of continuity. We now introduce a further condition on f_i which relates to the behavior that the function exhibits in a small neighborhood of a point x_i. The condition is formalized in the following definition. The phrase "locally monomodal" is coined to describe the subject property.

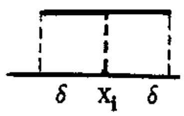
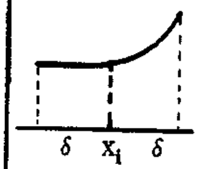
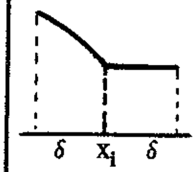
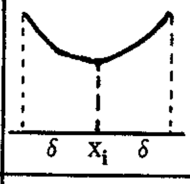
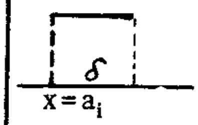
DEFINITION: A continuous function f_i over $[a_i,b_i]$ is said to be *locally monomodal* at point $x_i \in [a_i,b_i]$ if

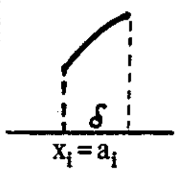
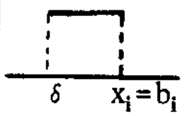
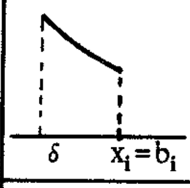
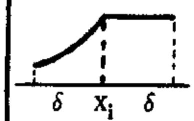
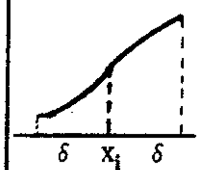
(i) there exists a $\delta(x_i)>0$ such that f_i is strictly increasing or strictly decreasing or constant on $[x_i,x_i+\delta]$ and strictly increasing or strictly decreasing or constant on $[x_i-\delta,x_i]$ whenever $x_i\in(a_i,b_i)$

(ii) there exists a $\delta(x_i)>0$ such that f_i is strictly increasing or strictly decreasing or constant on $[x_i,x_i+\delta]$ if $x_i=a_i$

(iii) there exists a $\delta(x_i)>0$ such that f_i is strictly increasing or strictly decreasing or constant on $[x_i-\delta,x_i]$ if $x_i=b_i$

The above definition describes a local characteristic of the function f_i in the sense that if the condition is satisfied for a certain $\delta_1(x_i)$, it is also satisfied for every $\delta_2(x_i)<\delta_1(x_i)$, and therefore one should examine a sufficiently small neighborhood to verify local monomodality at point x_i. Figure 2 shows an exhaustive compilation of the

Mode	1	2	3	4	5
Graph of f_i over $[x_i,x+\delta]$ and/or $[x_i-\delta,x_i]$					
Sign of Δf_i $\delta \geq \Delta x_i > 0$	0	+	0	+	0
Sign of Δf_i $-\delta \leq \Delta x_i < 0$	0	0	+	+	NA

Mode	6	7	8	9	10
Graph of f_i over $[x_i,x+\delta]$ and/or $[x_i-\delta,x_i]$					
Sign of Δf_i $\delta \geq \Delta x_i > 0$	+	NA	NA	0	+
Sign of Δf_i $-\delta \leq \Delta x_i < 0$	NA	0	+	–	–

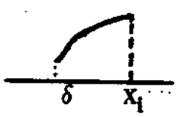
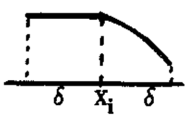
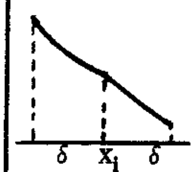
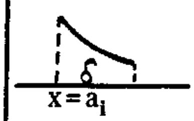
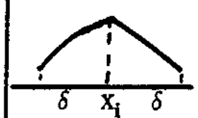
Mode	11	12	13	14	15
Graph of f_i over $[x_i,x+\delta]$ and/or $[x_i-\delta,x_i]$					
Sign of Δf_i $\delta \geq \Delta x_i > 0$	NA	–	–	–	–
Sign of Δf_i $-\delta \leq \Delta x_i < 0$	–	0	+	NA	–

FIG. 2. The 15 possible variation modes of a locally monomodal f_i at $x_i \in [a_i, b_i]$

15 possible modes of variational change that might be exhibited by a locally monomodal $f_i(x_i)$ at x_i. Each distinct mode represents a possible combination of strictly monotone or constant behavior on each side of the given point x_i if it is an internal point of $[a_i,b_i]$, or a possible behavior on one side if x_i is an end-point. Modes 1,2,3,4,9,10,12, 13,15 for internal x_i correspond to condition (i) of the definition, while modes 5,6,14 for $x_i=a_i$ and 7,8,11 for $x_i=b_i$ correspond respectively to conditions (ii) and (iii) of the definition. The figure tabulates for each mode the algebraic sign of the incremental change Δf_i in f_i for positive and negative incremental changes Δx_i in x_i. When x_i is an end-point of the interval $[a_i,b_i]$, the change Δx_i can be either positive or negative but not both. This is indicated by "NA" in the figure for the modes where the specified change Δx_i is not feasible.

It should be noted that the condition of local monomodality is a fairly mild restriction of little practical consequence since it is always satisfied in "real-world" situations. It excludes certain types of analytically pathological behavior, such as that exhibited by the function $f_i(x_i)=x_i\sin(1/x_i)$, which is continuous but not monomodal at the point $x=0$. This function exhibits an infinite number of local minima and maxima clustered on each side of $x_i=0$. Its behavior at $x_i=0$ cannot be identified as one of the 15 modes in Figure 2.

It can be shown that the *local* property of monomodality of the function $f_i(x_i)$ defined for a point $x_i \in [a_i,b_i]$ is equivalent to a *global* property of $f_i(x_i)$ over the entire interval $[a_i,b_i]$. One can show that $f_i(x_i)$ is locally monomodal at every point $x_i \in [a_i,b_i]$ if and only if the function f_i exhibits at most a finite number of local strict extrema and a finite number of intervals over which f_i is constant in the interval $[a_i,b_i]$. This equivalent global property may, in some cases, be easier to recognize and verify for a given f_i than testing the function f_i for monomodality at every $x_i \in [a_i,b_i]$. Note how this equivalence is exemplified by the function $f(x)=x \sin(1/x)$ which is not locally monomodal on any interval containing $x=0$ and exhibits an infinite number of local strict extrema over such intervals. We shall not concern ourselves here with the presentation of a formal proof for this equivalence between the two properties, since in this paper we shall refer only to the basic properties of locally monomodality as expressed in its definition stated above.

2.7 Modal Composition

In all the subsequent analysis of this paper, we shall assume that each given function $f_i(x_i)$ is locally monomodal over $[a_i,b_i]$. This means that for any given $x_i \in [a_i,b_i]$ the function $f_i(x_i)$ must exhibit one of the 15 distinct modes of variation tabulated in Figure 2. We introduce the notation $x_i(m)$ to denote that the function f_i exhibits mode number m at point x_i

$$x_i(m) \Leftrightarrow f_i \text{ exhibits mode } m \text{ at } x_i \, .$$

Furthermore, we examine the elements x_i of the sets $x = \{x_i\}$, X, and $\bar{X}$, defined above in (15) and (16), and use the same notation to denote the variation modes exhibited by the functions f_i at x_i. Let $x(m)$ represent the subset of x whose elements exhibit mode m

$$x(m) \equiv \{x_i(m) \in x\} \, .$$

Note that $x(m)$ may be empty signifying that x has no elements x_i of mode m, or may have any number of elements up to n. Extending the notation further, we write

$$x(m_1,m_2) \equiv x(m_1)+x(m_2) \equiv \{x_i(m_1) \in x \, , \, x_i(m_2) \in x\}$$

where "+" denotes the union operator for sets. This means $x(m_1,m_2)$ is the subset of x comprising all x_i of modes m_1 or m_2. Evidently one has the identity

$$x \equiv x(1,2,...,15) = x(1)+x(2)+...+x(15)$$

where some of the subsets x(m) may be empty. For example we may have

$$x = x(1,3,10) = x(1)+x(3)+x(10)$$

which implies that x comprises only elements x_i of modes 1, 3, and 10 and $x(m)=\phi$ for $m \neq 1,3,10$. Semantically we say the "modal composition" of x is (1,3,10). The same notation can be used in the same way to represent the modal composition of the sets X and $\bar{X}$. This can be illustrated by reference to Figure 3 which depicts ten monomodal functions and a given allocation $x=(x_1,x_2,\ldots,x_{10})$. The graphs of the functions are plotted only in the neighborhood of the points x_i to avoid undue cluttering of the figure. Comparing Figure 3 to Figure 2, we determine the modes of x_i as follows:

$$x_1(14),\ x_2(13),\ x_3(2),\ x_4(9),\ x_5(10),\ x_6(15),\ x_7(13),\ x_8(4),\ x_9(1),\ x_{10}(8)\ .$$

The mode is shown in the figure below each x_i. The modal compositions of x, X, $\bar{X}$ are

$$x = x(1,2,4,8,9,10,13,14,15)\ ,\quad X = X(4,8,9,10,13,14,15)\ ,\quad \bar{X} = \bar{X}(1,2,13)\ .$$

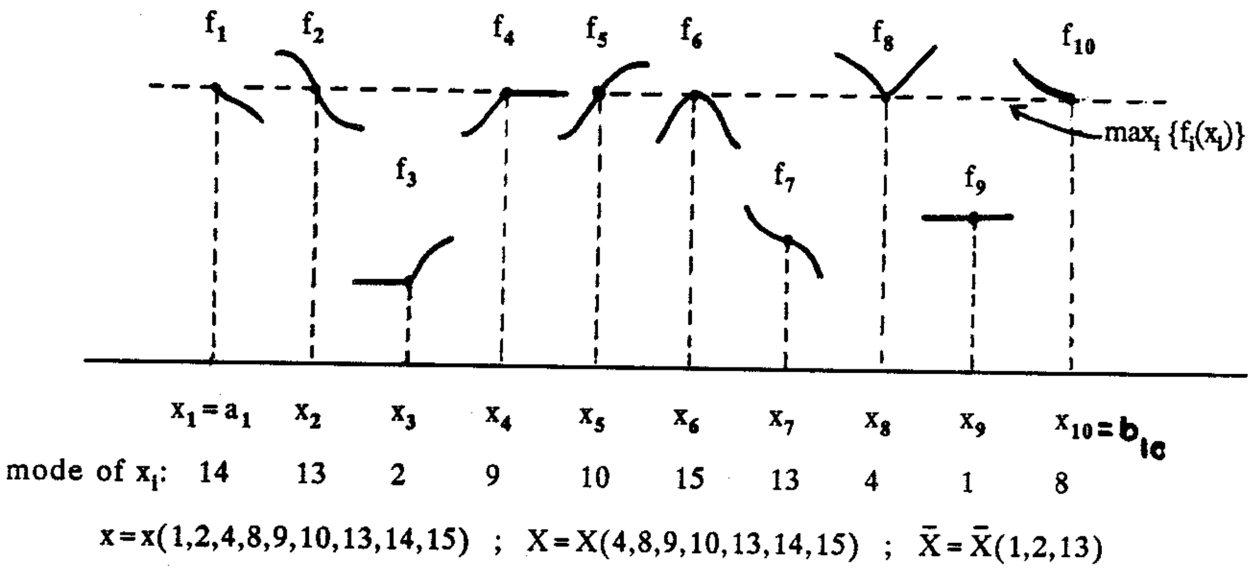


FIG. 3. Illustrating the modal composition of x, X, and $\bar{X}$

3. MAIN RESULTS

We now present the main results as four theorems stating the necessary and sufficient conditions for the local and global minimum and maximum points of the objective function $F(x) = \max_i f_i(x_i)$. We then show how these results are extendable in a straightforward fashion to the objective function $f(x) = \min_i f_i(x_i)$.

3.1 Local Minima

THEOREM 1

A point $x \in C(L,a,b)$ *is a local minimum point of* $F(x)$ *if and only if one of the following mutually exclusive conditions is satisfied:*

(C_1) $X(1,2,3,4,5,6,7,8)$ *is nonempty*

(C_2) $X = X(9,10,11)$ *and* $\bar{X} = \bar{X}(7,8,11)$

(C_3) $X = X(12,13,14)$ *and* $\bar{X} = \bar{X}(5,6,14)$.

PROOF

Sufficiency: Given a point $x \in C(L,a,b)$ for which one of the conditions C_1 or C_2 or C_3 is satisfied, we prove x is a local minimum point of F(x) by demonstrating the existence of a neighborhood $N_C(x,\delta)$ such that

$$F(x+\Delta x) \geq F(x) \quad \text{whenever } (x+\Delta x) \in N_C(x,\delta)$$

$$\Delta F(x,\Delta x) \geq 0 \quad \text{whenever } (x+\Delta x) \in N_C(x,\delta) \,. \tag{30}$$

We shall choose a sufficiently small $N_C(x,\delta)$, i.e., a sufficiently small $\delta > 0$, such that the conditions of the Lemma are satisfied. This can be done by choosing $\delta < \delta_0$ where δ_0 is the value whose existence is guaranteed by the Lemma. From the Lemma, the statement in (30) is equivalent to

$$\max_{i \in P} \Delta f_i(x_i,\Delta x_i) \geq 0 \quad \text{whenever } (x+\Delta x) \in N_C(x,\delta) \tag{31}$$

Since each function f_i is monomodal at x_i, there exists a δ_i such that f_i exhibits one of the 15 modes of variation shown in Figure 2 whenever $\Delta x_i \leq \delta_i$ (see definition of locally monomodal function). Again we choose $N_C(x,\delta)$ sufficiently small by choosing $\delta \leq \min\{\delta_i\}$ such that all the functions f_i exhibit the modes of variation in Figure 2 whenever $(x+\Delta x) \in N_C(x,\delta)$. We shall now show that the term $\max_{i\in P}\Delta f_i$ in (31) is nonnegative if C_1 or C_2 or C_3 is satisfied. To show that $\max_{i\in P}\Delta f_i$ is nonnegative, we have to demonstrate that there is at least one $j\in P$ for which $\Delta f_j \geq 0$. If C_1 is satisfied, there is at least one $j\in P$ such that f_j exhibits one of the modes 1 through 8 for which $\Delta f_j \geq 0$ (see Figure 2), and the required result follows.

From the definitions of $N_C(x,\delta)$ and C in (11) and (8), one has for all $(x+\Delta x) \in N_C(x,\delta)$:

$$L = \Sigma_{i\in(P+Q)}(x_i+\Delta x_i) = \Sigma_{i\in(P+Q)}x_i + \Sigma_{i\in(P+Q)}\Delta x_i = L + \Sigma_{i\in(P+Q)}\Delta x_i$$

$$\Sigma_{i\in(P+Q)}\Delta x_i = \Sigma_{i\in P}\Delta x_i + \Sigma_{i\in Q}\Delta x_i = 0 \qquad \text{for all } (x+\Delta x) \in N_C(x,\delta) . \qquad (32)$$

If condition C_2 is satisfied, $\bar{X}=\bar{X}(7,8,11)$ and therefore $\Delta x_i \leq 0$ for all $i\in Q$ (see modes 7, 8, and 11 in Figure 2). Thus $\Sigma_{i\in Q}\Delta x_i \leq 0$ and (32) implies $\Sigma_{i\in P}\Delta x_i \geq 0$ which implies $\Delta x_j \geq 0$ for some $j\in P$, and since $X=X(9,10,11)$ one has $\Delta f_j \geq 0$ and the required result follows. If condition C_3 is satisfied, $\bar{X}=\bar{X}(5,6,14)$ and therefore $\Delta x_i \geq 0$ for all $i\in Q$. Thus $\Sigma_{i\in Q}\Delta x_i \geq 0$ and (32) implies $\Sigma_{i\in P}\Delta x_i \leq 0$ which implies $\Delta x_j \leq 0$ for some $j\in P$, and since $X=X(12,13,14)$ one has $\Delta f_j \geq 0$ and the required result follows. Finally, we should note that if Q is empty, then (32) reduces to $\Sigma_{i\in P}\Delta x_i = 0$, which implies $\Delta x_j \geq 0$ and $\Delta x_k \leq 0$ for some $j,k\in P$, and the same conclusions under C_1 and C_2 still hold.

Necessity: Given point $x\in C$ is a local minimum point of F(x), we prove that one of the conditions C_1 or C_2 or C_3 must be true, i.e., we prove that C_0 must be true:

$$C_0 = C_1 + C_2 + C_3$$

where the "+" operator represents the logical OR. Note that each of C_1 and C_2 is the logical AND of two conditions. Let C_{21}, C_{22}, C_{31}, C_{32} represent these conditions:

$$C_{21} \Leftrightarrow X = X(9,10,11) \quad , \quad C_{22} \Leftrightarrow \bar{X} = \bar{X}(7,8,11) \tag{33}$$

$$C_{31} \Leftrightarrow X = X(12,13,14) \quad , \quad C_{32} \Leftrightarrow \bar{X} = \bar{X}(5,6,14) \tag{34}$$

$$C_0 = C_1 + C_{21}C_{22} + C_{31}C_{32} .$$

To prove the truth of C_0 by the method of contradiction, we assume C_0 is not true and arrive at a contradiction to the postulate that x is a local minimum point of F(x). Equivalently we assume NOT C_0 is true and arrive at the contradiction. By the familiar manipulation of Boolean algebra, NOT C_0 can be expressed as the logical OR of four alternative conditions.

$$\bar{C}_0 = \overline{C_1 + C_{21}C_{22} + C_{31}C_{32}} = \bar{C}_1(\overline{C_{21}C_{22}})(\overline{C_{31}C_{32}}) = \bar{C}_1(\bar{C}_{21} + \bar{C}_{22})(\bar{C}_{31} + \bar{C}_{32})$$

$$\bar{C}_0 = \bar{C}_1\bar{C}_{21}\bar{C}_{31} + \bar{C}_1\bar{C}_{21}\bar{C}_{32} + \bar{C}_1\bar{C}_{22}\bar{C}_{31} + \bar{C}_1\bar{C}_{22}\bar{C}_{32} . \tag{35}$$

Our objective is to show that if any of the four alternative conditions of $\bar{C}_0$ in (35) is true, a contradiction arises to the given fact that x is a local minimum point of F(x), i.e., given *any* δ^*-neighborhood $N_C(x,\delta^*)$ of x we can find a point $(x+\Delta x) \in N_C(x,\delta^*)$ such that

$$F(x+\Delta x) < F(x) \qquad \text{for some } (x+\Delta x) \in N_C(x,\delta^*)$$

$$\Delta F(x,\Delta x) < 0 \qquad \text{for some } (x+\Delta x) \in N_C(x,\delta^*) . \tag{36}$$

Consider a sufficiently small positive value $\delta < \delta^*$ and the corresponding neighborhood $N_C(x,\delta) \subset N_C(x,\delta^*)$ such that the conditions of the Lemma and Figure 2 are both satisfied. This is always possible since the functions f_i are monomodal at x_i. We shall show that we can specify a point $(x+\Delta x) \in N_C(x,\delta) \subset N_C(x,\delta^*)$ for which (36) is satisfied. From the Lemma the condition in (36) is equivalent to

$$\max_{i \in P} \Delta f_i(x_i + \Delta x_i) < 0 \qquad \text{for some } (x+\Delta x) \in N_C(x,\delta^*) .$$

Thus, for (36) to be satisfied we should choose the point $x+\Delta x$ such that

$$\Delta f_i < 0 \qquad \text{for all } i \in P . \tag{37}$$

This is satisfied if we choose each Δx_i such that the corresponding increment Δf_i is negative for all $i \in P$. The algebraic sign of Δf_i for a given Δx_i depends on the mode of f_i at x_i. Examine the modal composition of X:

$$X \equiv X(1,2, \ldots ,15) = X(1,2, \ldots ,8)+X(9,10,11)+X(12,13,14)+X(15). \qquad (38)$$

Each of the four alternative conditions in (35) includes the condition $\bar{C}_1$ which means $X(1,2, \ldots ,8) = \phi$, and (38) reduces to

$$X = X(9,10,11)+X(12,13,14)+X(15) . \qquad (39)$$

Examining the modes 9 through 15 in Figure 2, it is evident that (37) and (39) imply that the values of Δx_i should be chosen as follows:

$$\Delta x_i<0 \text{ for } x_i \in X(9,10,11); \ \Delta x_i>0 \text{ for } x_i \in X(12,13,14); \ \Delta x_i \neq 0 \text{ for } x_i \in X(15). \qquad (40)$$

What remains to be shown is that the values of Δx_i, chosen to satisfy (40), can also be made to satisfy the constraint $\Sigma_x \Delta x_i = 0$ which guarantees that $(x+\Delta x) \in N_C(x,\delta)$ as indicated by (32). Thus we should have

$$0 = \Sigma_x \Delta x_i = \Sigma_{X+\bar{X}} \Delta x_i = \Sigma_{X(9,10,11)} \Delta x_i + \Sigma_{X(12,13,14)} \Delta x_i + \Sigma_{X(15)} \Delta x_i + \Sigma_{\bar{X}} \Delta x_i. \qquad (41)$$

According to the condition $\Delta x_i \neq 0$ for $x_i \in X(15)$ in (40), each of the values of Δx_i in the third summation on the right hand side of (41) may be chosen to be either positive or negative. Accordingly the set X(15) may be expressed as the union of two subsets $X(15) = X^+(15)+X^-(15)$ where $X^+(15)$ comprises the elements x_i for which Δx_i is chosen to be positive and $X^-(15)$ comprises the elements x_i for which Δx_i is chosen negative. The requirement in (41) may therefore be rewritten as:

$$\Sigma_{X(9,10,11)} \Delta x_i + \Sigma_{X(12,13,14)} \Delta x_i + \Sigma_{X^+(15)} \Delta x_i + \Sigma_{X^-(15)} \Delta x_i + \Sigma_{\bar{X}} \Delta x_i = 0 . \qquad (42)$$

For convenience, we choose $|\Delta x_i|$ to have the same value within each of the first four summations in (42) which may then be expressed as:

$$- n_1 d_1 + n_2 d_2 + n_3 d_3 - n_4 d_4 + \Sigma_{\bar{X}} \Delta x_i = 0 \ ; \ d_1, d_2, d_3, d_4 > 0 \qquad (43)$$

where n_1, n_2, n_3, n_4 represent the number of elements in the respective sets X(9,10,11), X(12,13,14), $X^+(15)$, and $X^-(15)$, and d_1, d_2, d_3, d_4 are the common values of $|\Delta x_i|$ within each summation. The signs of the terms in (43) reflect the conditions on Δx_i in (40). It should be noted that in (43) the values n_1, n_2 and the number of elements in X(15), denoted by $|X(15)| = n_3 + n_4$, are fixed parameters determined by the given point x, while d_1, d_2, d_3, d_4 and Δx_i (in the summation) can be considered as variables (arbitrarily small) whose values are chosen so that (43) is satisfied. Note also that the values of n_3 and n_4 can be arbitrarily chosen subject to condition $n_3 + n_4 = |X(15)|$.

We shall now show that if any of the four alternative conditions in (35) is true, it is always possible to satisfy the requirement in (43) by appropriate selection of the values of nonfixed parameters.

(A) Assume the first alternative condition $\bar{C}_1\bar{C}_{21}\bar{C}_{31}$ in (35) is true. From (33) and (34) we obtain

$$X(1,2,\ldots,8) = \phi \ , \quad X \neq X(9,10,11) \ , \quad X \neq X(12,13,14) \ . \tag{44}$$

Recall the identity

$$X \equiv X(1,2,\ldots,15) = X(1,2,\ldots,8) + X(9,10,11) + X(12,13,14) + X(15). \tag{45}$$

Combining (44) and (45) we obtain

$$X(12,13,14) + X(15) \neq \phi \ , \quad X(9,10,11) + X(15) \neq \phi \ . \tag{46}$$

The statement in (46) is equivalent to either

(A1) $X(15) \neq \phi$

(A2) $X(9,10,11) \neq \phi$ and $X(12,13,14) \neq \phi$.

If (A1) is true then $|X(15)| = n_3 + n_4 \neq 0$, in which case we rewrite the requirement in (43) as

$$n_3 d_3 - n_4 d_4 = n_1 d_1 - n_2 d_2 - \Sigma_{\bar{X}} \Delta x_i \tag{47}$$

which can always be satisfied as follows: if the right hand side is positive, choose $n_4=0$ and $n_3=|X(15)|\neq 0$; if negative, choose $n_3=0$ and $n_4=|X(15)|\neq 0$. The right hand side can be zero for a *specific choice* of the values d_1, d_2, and Δx_i, in which case decrement any one of these parameters by an arbitrarily small amount to make the right hand side positive or negative and proceed as before. On the other hand, the right hand side may be *identically* equal to zero if $n_1=n_2=|\bar{X}|=0$, in which case $x=X=X(15)$ and $|X(15)|=|x|=n\geq 2$. This means we can choose $n_3\geq 1$ and $n_4\geq 1$ and then choose d_3 and d_4 such that $n_3d_3-n_4d_4=0$.

If (A2) is true, then $n_1\neq 0$ and $n_2\neq 0$ and the requirement in (43) may be rewritten as:

$$n_1d_1 - n_2d_2 = n_3d_3 - n_4d_4+\Sigma_{\bar{X}}\Delta x_i \tag{48}$$

which can always be satisfied by choosing $(d_1/d_2) > (n_2/n_1)$ if the right hand side is positive, $(d_1/d_2) < (n_2/n_1)$ if the right hand side is negative, and $(d_1/d_2) = (n_2/n_1)$ if the right hand side is zero.

(B) Assume the second alternative condition $\bar{C}_1\bar{C}_{21}\bar{C}_{32}$ in (35) is true. From (33) and (34) we obtain

$$X(1,2,\ldots,8) = \phi \ , \ X \neq X(9,10,11) \ , \ \bar{X} \neq \bar{X}(5,6,14) \ . \tag{49}$$

Combining (45) and (49) we obtain

$$X(12,13,14)+X(15) \neq \phi \ .$$

Hence either of the following two conditions (B1) or (B2) must be true

(B1) $X(15)\neq\phi$, which is identical to (A1) above and the proof is the same, or

(B2) $X(12,13,14)\neq\phi$, which means $n_2\neq 0$, in which case we rewrite the requirement in (43) as

$$n_2d_2 = n_1d_1+n_4d_4 - n_3d_3 - \Sigma_{\bar{X}}\ \Delta x_i > 0 \ . \tag{50}$$

This can always be satisfied as follows: if either $n_1 \neq 0$ or $|X(15)| \neq 0$, choose $n_3 = 0$ and $\Delta x_i = 0$; but if both $n_1 = 0$ and $|X(15)| = 0$, then $n_3 = n_4 = 0$ and (50) reduces to

$$n_2 d_2 = -\Sigma_{\bar{X}} \Delta x_i > 0 . \tag{51}$$

We now examine the last condition in (49), namely $\bar{X} \neq \bar{X}(5,6,14)$, which implies that there is at least one value $x_j \in \bar{X}$ such that $x_j \neq a_j$. Otherwise we would have $\bar{X} = \bar{X}(5,6,14)$, as can be verified from Figure 2. To satisfy (51), choose $\Delta x_i = 0$ for all $i \neq j$ and $\Delta x_j < 0$ which is possible since $x_j \neq a_j$. (Note that the condition $\bar{X} \neq \bar{X}(5,6,14)$ implies $\bar{X} \neq \phi$, for if $X = \phi$ then $\bar{X}(5,6,14) = \phi$ and $X = \bar{X}(5,6,14)$ is always satisfied.)

(C) Assume the third alternative condition $\bar{C}_1 \bar{C}_{22} \bar{C}_{31}$ in (35) is true. The proof is analogous to the proof in (B) with n_1 replacing the role of n_2 and b_j replacing the role of a_j.

(D) Finally, assume the fourth alternative condition $\bar{C}_1 \bar{C}_{22} \bar{C}_{32}$ in (35) is true. From (33) and (34) we obtain

$$X(1,2, \ldots ,8) = \phi \ , \ \bar{X} \neq \bar{X}(7,8,11) \ , \ \bar{X} \neq \bar{X}(5,6,14) . \tag{52}$$

The second condition in (52) implies that there is at least one $x_j \in \bar{X}$ such that $x_j \neq a_j$, and the third condition implies that there is at least one $x_k \in \bar{X}$ such that $x_k \neq b_k$. The requirement in (43) is now rewritten as

$$\Sigma_{\bar{X}} \Delta x_i = n_1 d_1 - n_2 d_2 - n_3 d_3 + n_4 d_4$$

which can always be satisfied as follows: if the right hand side is positive, choose $\Delta x_i = 0$ for all $i \neq k$ and $\Delta x_k > 0$ which is feasible since $x_k \neq b_k$; if the right hand side is negative, choose $\Delta x_i = 0$ for all $i \neq j$ and $\Delta x_j < 0$ which is feasible since $x_j \neq a_j$; if the right hand side is zero, choose $\Delta x_i = 0$ for all $x_i \in \bar{X}$.

We have proven that if $\bar{C}_0$ in (35) is true, we can produce a point $(x + \Delta x)$ satisfying (36), which means $(x + \Delta x) \in N_C(x,\delta)$ and $F(x + \Delta x) < F(x)$, which contradicts the starting assumption that x is a local minimum point of F(x). Hence $\bar{C}_0$ is not true, i.e., $C_0=C_1+C_2+C_3$ is true, and one of the conditions C_1 or C_2 or C_3 must be satisfied. This completes the necessity part, and the entire proof, of Theorem 1.

3.2 Local Maxima

THEOREM 2

A given $x \in C(L,a,b)$ *is a local maximum point of* F(x) *if and only if*

$$X = X(1,5,7,9,11,12,14,15)\ . \qquad (53)$$

PROOF

Sufficiency: Given a point $x \in C(L,a,b)$ for which the condition (53) is satisfied, we prove x is a local maximum point of F(x) by demonstrating the existence of a neighborhood $N_C(x,\delta)$ such that

$$F(x+\Delta x) \leq F(x) \qquad \text{for all } (x+\Delta x) \in N_C(x,\delta)$$

$$\Delta F(x,\Delta x) \leq 0 \qquad \text{for all } (x+\Delta x) \in N_C(x,\delta)\ . \qquad (54)$$

We choose a sufficiently small $N_C(x,\delta)$ such that the conditions of the Lemma and Figure 2 are satisfied. From the Lemma, the requirement in (54) is equivalent to

$$\max_{i \in P} \Delta f_i(x_i+\Delta x_i) \leq 0 \qquad \text{for all } \|x\| < \delta$$

$$\Delta f_i(x_i+\Delta x_i) \leq 0 \qquad \text{for all } i \in P$$

which is evidently true from Figure 2, since $\Delta f_i \leq 0$ for $x_i \in X$ exhibiting modes 1,5,7,9,11, 12,14,15.

Necessity: Given $x \in C$ is a local maximum point of F(x), we prove that (53) must be true. We shall assume, to the contrary, that (53) is not true and arrive at a contradiction:

$$X \neq X(1,5,7,9,11,12,14,15)\ . \qquad (55)$$

The assumption in (55) implies that

$$X(2,3,4,6,8,10,13) \neq \phi\ . \qquad (56)$$

We shall show that (56) leads to a contradiction to the stipulation of x being a local maximum point, by demonstrating that for *any* neighborhood $N_C(x,\delta^*)$ of x we can find a point $(x+\Delta x)\in N_C(x,\delta^*)$ such that

$$F(x+\Delta x) > F(x) \qquad \text{for some } (x+\Delta x)\in N_C(x,\delta^*) \tag{57}$$

$$\Delta F(x,\Delta x) > 0 \qquad \text{for some } (x+\Delta x)\in N_C(x,\delta^*)\ . \tag{58}$$

Consider a sufficiently small value $\delta<\delta^*$ and the corresponding neighborhood $N_C(x,\delta) \subset N_C(x,\delta^*)$ such that the conditions of the Lemma and Figure 2 are both satisfied. We shall prove that we can specify a point $(x+\Delta x)\in N_C(x,\delta)\subset N_C(x,\delta^*)$ such that (58) is satisfied. By the Lemma, (58) becomes

$$\max_{i\in P}\Delta f_i(x_i,\Delta x_i) > 0 \qquad \text{for some } (x+\Delta x)\in N_C(x,\delta^*)\ . \tag{59}$$

Thus, for (59) to be satisfied we should choose the $x+\Delta x$ such that

$$\Delta f_i(x_i,\Delta x_i) > 0 \qquad \text{for some } j \in P\ . \tag{60}$$

Since $j\in P$ means $x_j\in X$, the last requirement in (60) becomes

$$\Delta f_j(x_j,\Delta x_j) > 0 \qquad \text{for some } x_j \in X \text{ and some } \Delta x_j\ . \tag{61}$$

In choosing the required point $x+\Delta x$, we shall, for the sake of simplicity, restrict the increment vector Δx to have only two nonzero elements: Δx_j corresponding to x_j and Δx_k corresponding to some other x_k to be specified later; thus

$$\Delta x = (0,0,\ldots,\Delta x_j,0,\ldots,\Delta x_k,0,\ldots,0)\ .$$

Since $\Sigma_x \Delta x_i = 0$, we should have

$$\Delta x_k = -\Delta x_j\ .$$

The assumption (56) can be written as

$$X(2,4,6,10) + X(3,8,13) \neq \phi$$

which implies either $x(2,4,6,10)\neq\phi$ or $X(3,8,13)\neq\phi$.

(1) If $X(2,4,6,10)\neq\phi$, then choose $x_j\in X(2,4,6,10)$ and $\Delta x_j>0$, which makes $\Delta f_j>0$, as shown in Figure 2, and (61) is satisfied. This means we must choose $\Delta x_k<0$, which

is always feasible if we choose

$$x_k \neq a_k \quad \text{some } k \neq j . \tag{62}$$

We contend that under the present conditions such an x_k, as indicated in (62), does exist, because if otherwise $x_k = a_k$ for all $k \neq j$, then we would have

$$x_j = L - \Sigma_{k\neq j} x_k = L - \Sigma_{k\neq j} a_k$$

which means $x_j = b_j$ (since x_j cannot have any value larger than $L - \Sigma_{k\neq j} a_k$), which is a contradiction to $x_j \in X(2,4,6,10)$.

(2) If $X(3,8,13) \neq \phi$, then choose $x_j \in X(3,8,13)$ and $\Delta x_j < 0$, which makes $\Delta f_j > 0$, as shown in Figure xx, and () is satisfied. This means we must choose $\Delta x_k > 0$, which is always feasible if we choose

$$x_k \neq b_k \quad \text{some } k \neq j . \tag{63}$$

Under the present conditions such an x_k, as indicated in (63), does exist, because if otherwise $x_k = b_k$ for all $k \neq j$, then we would have

$$x_j = L - \Sigma_{k\neq j} x_k = L - \Sigma_{k\neq j} b_k$$

which means $x_j = a_j$ (since x_j cannot have any value smaller than $L - \Sigma_{k\neq j} b_k$), which is a contradiction to $x_j \in X(3,8,13)$.

This completes the proof of Theorem 2. We next direct our attention to the global minima and maxima of F(x).

3.3 Global Minimum Points

THEOREM 3

Let $\check{X}_m$ *be the set of all local minimum points and* $\check{X}_g$ *be the set of all global minimum points of* F(x) *in* C(L,a,b); *then*

(i) $\check{X}_g \subseteq \check{X}_m$

(ii) $z \in \check{X}_g$ *if and only if* $F(z) = \min_{x \in \check{X}_m} F(x)$.

PROOF

The set C(L,a,b) is the intersection of the compact n-cell defined by $x_i \in [a_i, b_i]$ and the closed set of the hyperplane defined by $x_1 + x_2 + \ldots + x_n = L$. Hence C is compact. By the Lemma, F(x) is a continuous function over C and must therefore attain its global minimum value F for some point $x \in C$

$$F \equiv \min_{x\in C} F(x) = F(x) \qquad x \in C . \tag{64}$$

We now show that $\breve{X}_g \subseteq \breve{X}_m$. Consider any point $z \in \breve{X}_g$ and a δ-neighborhood $N_C(z,\delta)$. Since z is a global minimum point, one has $F(z+\Delta z) \geq F(z)$ for all $(z+\Delta z) \in C$, and since $N_C \subseteq C$, we have

$$F(z+\Delta z) \geq F(z) \qquad \text{for all } (z+\Delta z) \in N_C(z,\delta) . \tag{65}$$

The statement in (65) makes z a local minimum point, i.e., $z \in \breve{X}_m$, hence $\breve{X}_g \subseteq \breve{X}_m$. We now prove the second part of the Theorem.

Sufficiency: Given a point $z \in C$ such that

$$F(z) = \min_{x\in \breve{X}m} F(x) . \tag{66}$$

We shall prove that $z \in \breve{X}_g$. Since $\breve{X}_g \subseteq \breve{X}_m$, one has

$$\min_{x\in \breve{X}m} F(x) \leq \min_{x\in \breve{X}g} F(x) . \tag{67}$$

The left hand side is equal to F(z) and the right side is F, hence

$$F(z) \leq F . \tag{68}$$

Similarly, since $\breve{X}_m \subseteq C$, one has

$$\min_{x\in C} F(x) \leq \min_{x\in \breve{X}m} F(x) . \tag{69}$$

The left side is F and the right side is F(z) as indicated by (64) and (66); hence

$$F \leq F(z) . \tag{70}$$

From (68) and (70) we obtain $F(z) = F$, i.e., z is a global minimum point of F(x) and $z \in \breve{X}_g$.

Necessity: Let z be a global minimum point of F(x), i.e., $z \in \breve{X}_g$,

$$F(z) = F . \tag{71}$$

We shall prove that $F(z)=\min_{x\in\check{X}m}F(x)$. Assume to the contrary that

$$F(z) \neq \min_{x\in\check{X}m}F(x)\ . \tag{72}$$

The statement in (72) implies

$$F(z) = F < \min_{x\in\check{X}m}F(x)\ . \tag{73}$$

Note that the other alternative, $F>\min_{x\in\check{X}m}F(x)$, is impossible because F is the global minimum value of F(x). We have already shown that $\check{X}_g\subseteq\check{X}_m$, hence

$$\min_{x\in\check{X}m}F(x) \leq \min_{x\in\check{X}g}F(x) = F$$

which is a contradiction to (73). Hence, the assumption in (72) cannot be true, and $F(z)=\min_{x\in\check{X}m}F(x)$.

3.4 Global Maximum Points

THEOREM 4

Let $\hat{X}_m$ *be the set of all local maximum points and* $\hat{X}_g$ *be the set of all global maximum points of* F(x) *in* C(L,a,b); *then*

(i) $\hat{X}_g \subseteq \hat{X}_m$

(ii) $z \in \hat{X}_g$ *if and only if* $F(z)=\max_{x\in\hat{X}m}F(x)$.

PROOF

We have already shown in the proof of Theorem 3 that C is compact. By the Lemma, F(x) is a continuous function over C and must therefore attain its global maximum value F for some point $x\in C$

$$F \equiv \max_{x\in C}F(x) = F(x) \qquad x \in C\ . \tag{74}$$

We now show that $\hat{X}_g\subseteq\hat{X}_m$. Consider any point $z\in\hat{X}_g$ and a δ-neighborhood $N_C(z,\delta)$. Since z is a global maximum point, one has $F(z+\Delta z)\leq F(z)$ for all $(z+\Delta z)\in C$, and since $N_C\subseteq C$, we have

$$F(z+\Delta z) \leq F(z) \qquad \text{for all } (z+\Delta z) \in N_C(z,\delta)\ . \tag{75}$$

The statement in (75) makes z a local maximum point, i.e., $z \in \hat{X}_m$, hence $\hat{X}_g \subseteq \hat{X}_m$. We now prove the second part of the Theorem.

Sufficiency: Given a point $z \in C$ such that

$$F(z) = \max_{x \in \hat{X}_m} F(x) \ . \tag{76}$$

We shall prove that $z \in \hat{X}_g$. Since $\hat{X}_g \subseteq \hat{X}_m$, one has

$$\max_{x \in \hat{X}_m} F(x) \geq \max_{x \in \hat{X}_g} F(x) \ . \tag{77}$$

The left hand side is equal to F(z) and the right side is F, hence

$$F(z) \geq F \ . \tag{78}$$

Similarly, since $\hat{X}_m \subseteq C$, one has

$$\max_{x \in C} F(x) \geq \max_{x \in \hat{X}_m} F(x) \ . \tag{79}$$

The left side is F and the right side is F(z) as indicated by (74) and (76); hence

$$F \geq F(z) \ . \tag{80}$$

From (78) and (80) we obtain $F(z) = F$, i.e., z is a global maximum point of F(x) and $z \in \hat{X}_g$.

Necessity: Let z be a global maximum point of F(x), i.e., $z \in \hat{X}_g$,

$$F(z) = F \ . \tag{81}$$

We shall prove that $F(z) = \max_{x \in \hat{X}_m} F(x)$. Assume to the contrary that

$$F(z) \neq \max_{x \in \hat{X}_m} F(x) \ . \tag{82}$$

The statement in (82) implies

$$F(z) = F < \max_{x \in \hat{X}_m} F(x) \ . \tag{83}$$

Note that the other alternative, $F < \max_{x \in \hat{X}_m} F(x)$, is impossible because F is the global maximum value of F(x). We have already shown that $\hat{X}_g \subseteq \hat{X}_m$; hence

$$\max_{x \in \hat{X}_m} F(x) \geq \max_{x \in \hat{X}_g} F(x) = F$$

which is a contradiction to (83). Hence, the assumption in (82) cannot be true, and $F(z) = \max_{x \in \hat{X}_m} F(x)$.

3.5 Extensions to $f(x) \equiv \min_i f_i(x_i)$

All the foregoing results, which were derived for the resource allocation problem with objective function $F(x) = \max_i f_i(x_i)$, are readily extendable to the same problem with objective function $f(x) = \min_i f_i(x_i)$. The analytical connection between the two problems derives from the simple general identities

$$\min_i f_i(x_i) \equiv -\max_i[-f_i(x_i)] \tag{84}$$

$$\min_{x\in C} f(x) \equiv -\max_{x\in C}[-f(x)] \tag{85}$$

$$\max_{x\in C} f(x) \equiv -\min_{x\in C}[-f(x)] \ . \tag{86}$$

The min and max in (85) and (86) signify the absolute minimum and absolute maximum values of the functions $f(x)$ and $-f(x)$. These relations are also true of the local minima and maxima, a fact we represent by the following notation

$$\mathrm{lmin}_{x\in C} f(x) \equiv -\mathrm{lmax}_{x\in C}[-f(x)] \tag{87}$$

$$\mathrm{lmax}_{x\in C} f(x) \equiv -\mathrm{lmin}_{x\in C}[-f(x)] \tag{88}$$

where (87) and (88) are to be read: "for every $x\in C$ at which $f(x)$ exhibits a local minimum (maximum) value, $-f(x)$ exhibits a local maximum (mimimum) value of equal magnitude but opposite sign." Referring back to (84), let $g_i(x_i)$ denote $-f_i(x_i)$ and $G(x)$ denote $\max_i g_i(x_i)$; thus

$$f(x) = \min_i f_i(x_i) = -\max_i[-f_i(x_i)] = -\max_i g_i(x) = -G(x) \ . \tag{89}$$

Note that the function $G(x) = \max_i g_i(x)$ has the same form as the objective function $F(x) = \max_i f_i(x_i)$ for which all the previous results in the Lemma and the four theorems were derived. Thus these results can be used to find all the local and global minima and maxima of $G(x)$. Since (89) indicates that $G(x) = -f(x)$, the statements in (85), (86), (87), (88) indicate that the local minimum (maximum) points found for $G(x)$ are the local maximum (minimum) points of $f(x)$. The same is true for the global minimum and maximum points.

4. EXAMPLES

We now illustrate the foregoing concepts and criteria via simple examples with $n=2$. The two functions $f_1(x_1)$ and $f_2(x_2)$ are specified by their graphs in Figure 4. We have deliberately elected to represent the functions $f_i(x_i)$ by their graphs rather than by their analytical expressions to emphasize the fact that the results of this paper are equally applicable in real-world optimization problems where the given functions $f_i(x_i)$ are determined from empirical or simulation data which can be plotted into cartesian graphs for which no exact analytical expressions can be specified. Figures 5a,b,c,d,e show plots of the function $F(x)=\max_i f_i(x_i)$ for $L=2,4,6,8,10$ respectively, with $a_i=0$ and $b_i=L$ in all cases. Note that, as illustrated in Figure 5e, a point on the horizontal axis of these graphs represents the 2-tuple (x_1,x_2) with $x_1+x_2=L$, x_1 and x_2 being measured from the left and right ends of the interval $[0,L]$ respectively.

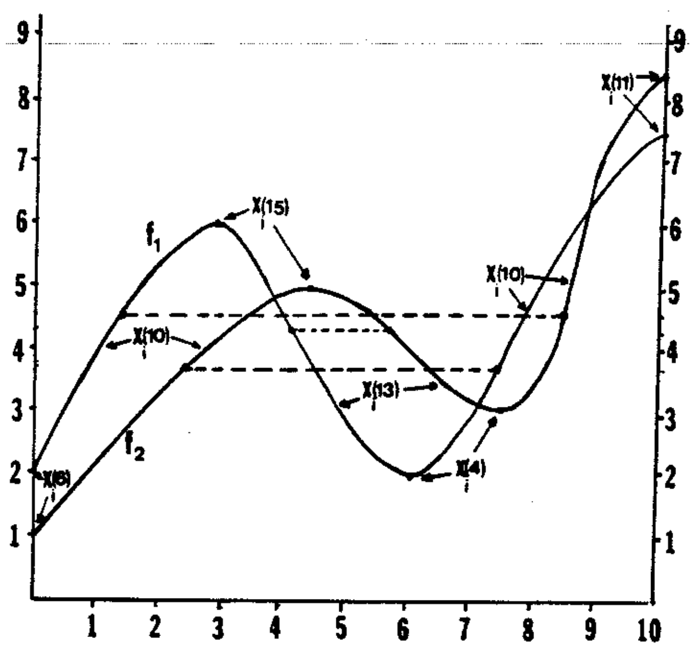


FIG. 4. Graphs and modal composition of the functions $f_1(x_1)$ and $f_2(x_2)$ used in the examples.

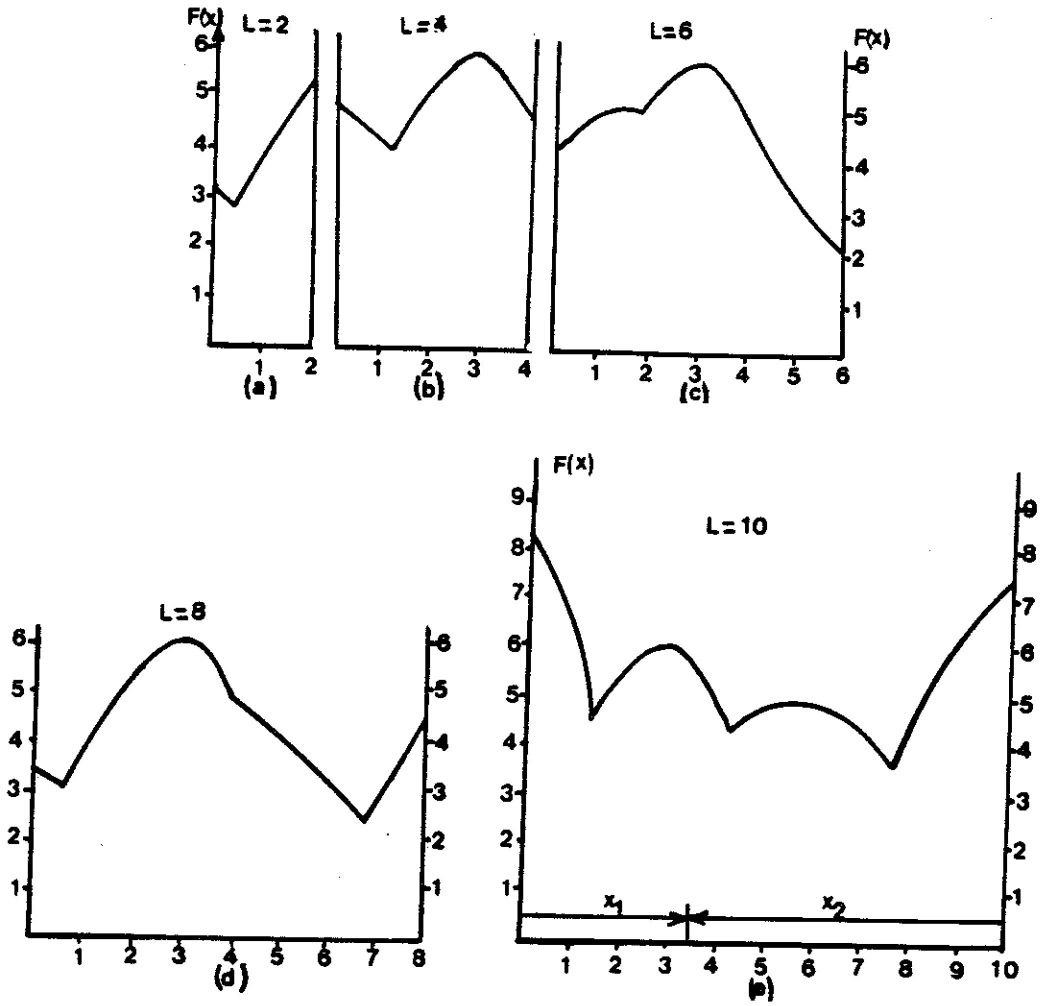


FIG. 5. Maxima and minima of F(x) of the example for various values of L

We now explain the details of applying the four theorems to find all the local and global extremum points for the case L = 10 shown in Figure 5e. Note that by inspecting the graphs of $f_1(x_1)$ and $f_2(x_2)$ we determine the following:

$$\text{modes exhibited by } x_1 : 4,6,10,11,13,15 \tag{90}$$

$$\text{modes exhibited by } x_2 : 4,6,10,11,13,15\ . \tag{91}$$

The specific sets of points $x_i(m)$ exhibiting mode m are indicated on Figure 4. Recall the meaning of X, $\bar{X}$, and $X(m_1,m_2,...,m_k)$ from (15), (16), and section 2.7 respectively:

$$X = \{x_i \in x : f_i(x_i) = \max_i f_i(x_i) \equiv F(x)\} \tag{92}$$

$$\bar{X} = x - X = \{x_i \in x : f_i(x_i) < \max_i f_i(x_i) \equiv F(x)\} \tag{93}$$

$$X(m_1,m_2,...,m_k) = \{x_i \in X : \text{mode of } x_i = m_1,m_2,...,m_k\} . \tag{94}$$

First we determine the set of local minimum points (lminp) satisfying condition C_1 of Theorem 1, which we shall denote by $\breve{X}_m(C_1)$. Let $x=(x_1,x_2)$ be a point satisfying C_1:

$$X(1,2,...,8) \neq \phi . \tag{95}$$

From (90) and (91) we have $X(1,2,...,8)=X(4,6)=X(4)+X(6)$. Thus condition (95) implies

$$X(4)+X(6) \neq \phi \quad , \quad X(4) \neq \phi \text{ or } X(6) \neq \phi .$$

Assume first $x_1 \in X(4) \subseteq X$. There is only one such point, namely $x_1=6$, which exhibits mode 4 with $f_1(6)=2$. Hence $x_2=L-x_1=10-6=4$, $f_2(4)=4.9$ and $f_1(6)<f_2(4)$ and $x_1 \notin X$, which is a contradiction. Next assume $x_2 \in X(4)$; hence $x_2=7.5$, $f_2(7.5)=3$, $x_1=10-7.5=2.5$, $f_1(2.5)=5.85$, $f_2(7.5)<f_1(2.5)$, $x_2 \notin X$, a contradiction. Next assume $x_1 \in X(6)$; hence $x_1=0$, $f_1(0)=2$, $x_2=10$, $f_2(10)=8.35$, $f_1(0)<f_2(10)$, $x_1 \in X$, a contradiction. Finally assume $x_2 \in X(6)$; hence $x_2=0$, $f_2(0)=1$, $x_1=10$, $f_1(10)=7.95$, $f_2(0)<f_1(10)$, $x_2 \notin X$, a contradiction. Thus in all cases (95) leads to a contradiction; therefore there can be no point x that satisfies (95) and $\breve{X}_m(C_1)$ is empty.

Next we determine the set $\breve{X}_m(C_2)$ of lminp satisfying C_2. Let $x=(x_1,x_2)$ be a point satisfying C_2:

$$X = X(9,10,11) \quad , \quad \bar{X} = \bar{X}(7,8,11) . \tag{96}$$

From (90) and (91), the conditions in (96) become

$$X = X(10,11) \quad , \quad \bar{X} = \bar{X}(11) . \tag{97}$$

Assume first that $\bar{X} \neq \phi$; hence $\bar{X}(11) \neq \phi$. If $x_1 \in \bar{X}(11)$, then $x_1 = 10$, $f_1(11) = 7.5$, $x_2 = 0$, $f_2(0) = 1$, $f_2(x_2) < f_1(x_1)$, $x_1 \in X$, $x_1 \notin \bar{X}$, a contradiction. Similarly, if $x_2 \in \bar{X}(11)$, then $x_2 = 10$, $f_2(10) = 8.35$, $x_1 = 0$, $f_1(0) = 2$, $f_1(x_1) < f_2(x_2)$, $x_2 \in X$, $x_2 \notin \bar{X}$, a contradiction. Thus the assumption $\bar{X} \neq \phi$ always leads to a contradiction, hence $\bar{X} = \phi$ and $\{x_1, x_2\} = X$. Thus $f(x_1) = f(x_2) = F(x)$. From (97) we have

$$\{x_1, x_2\} = X = X(10,11) = X(10) + X(11) \quad , \quad \bar{X} = \phi . \tag{98}$$

We now show that neither x_1 nor x_2 can belong to X(11). If $x_1 \in X(11)$, then $x_1 = 10$, $x_2 = 0$, $f(x_2) < f(x_1)$, $x_2 \in \bar{X}$, which contradicts $\bar{X} = \phi$. If $x_2 \in X(11)$, $x_2 = 10$, $x_1 = 0$, $f_1(x_1) < f_2(x_2)$, $x_1 \in \bar{X}$, which contradicts $\bar{X} = \phi$. Thus (98) becomes

$$\{x_1, x_2\} = X(10) . \tag{99}$$

This means both x_1 and x_2 must belong to the intervals where f_1 and f_2 are increasing, with $f_1(x_1) = f_2(x_2)$ and $x_1 + x_2 = 10$. The values of x_1 and x_2 that satisfy these requirements can be easily searched for by sliding a horizontal line, representing $F(x) = f_1(x_1) = f_2(x)$, in a vertical direction and summing up its x_1 and x_2 intercepts with the *increasing* portions of f_1 and f_2 and determining the values of F(x) for which $x_1 + x_2 = 10$. Two such points are found:

$$F(x) = 4.5 \text{ with } x = (1.3,8.7) \quad \text{and} \quad F(x) = 3.7 \text{ with } x = (7.65,2.35) . \tag{100}$$

The conditions represented by the points in (100) are depicted graphically in Figure 4. The top dotted line corresponds to the first point and the bottom dotted line corresponds to the second point. Thus

$$\breve{X}_m(C_2) = \{(1.3,8.7) , (7.65,2.35)\} \quad , \quad F(1.3,8.7) = 4.5 \quad , \quad F(7.65,2.35) = 3.7 . \tag{101}$$

The set $\breve{X}_m(C_3)$ of lminp satisfying C_3 is determined in an analogous fashion. We find $\bar{X} = \phi$ and $\{x_1, x_2\} = X = X(13)$. Hence x can be found by sliding a horizontal line in a vertical direction and summing up its x_1 and x_2 intercepts with the *decreasing* portions of f_1 and f_2 and determining the values of F(x) for which $x_1+x_2=10$. In this case only one such point is found, as represented by the middle dotted line in Figure 4:

$$F(x)=4.25 \quad , \quad \breve{X}_m(C_3)=(4.25,5.75) \ . \tag{102}$$

The set of all lminp is the union of $\breve{X}_m(C_1)$, $\breve{X}_m(C_2)$, and $\breve{X}_m(C_3)$

$$\breve{X}_m = \{(1.3,8.7) \ , \ (4.25,5.75) \ , \ (7.65,2.35)\} \ .$$

The global minimum point, by Theorem 3, is the lminp with the smallest value of F(x)

$$F(7.65,2.35)=4.25 \ .$$

Next, we turn our attention to finding the set $\hat{X}_m$ of all local maxima by Theorem 2. Let $x=(x_1,x_2)$ be a local maximum point (lmaxp). From (53), (90), and (91) we have

$$X = X(1,5,7,9,11,12,14,15) = X(11,15) = X(11)+X(15) \ .$$

Since $X \neq \phi$, we have either $x_1 \in X$ or $x_2 \in X$. If $x_1 \in X(11)$, then $x_1=10$, $f_1(x_1)=7.4$, $x_2=10-10=0$, $f_2(0)=1$, $f_2(x_2)<f_1(x_1)$; hence $x=(10,0)$ is a lmaxp with $F(10,0)=7.4$. If $x_2 \in X(11)$, then $x_2=10$, $f_2(10)=8.25$, $x_1=0$, $f_1(0)=2$, $f_1(0)<f_2(0)$; hence (0,10) is a lmaxp with $F(0,10)=8.25$. If $x_1 \in X(15)$, then $x_1=3$, $f_1(3)=6$, $x_2=7$, $f_2(7)=3.1$, $f_2(7)<f_1(3)$; hence (3,7) is a lmaxp with $F(3,7)=6$. If $x_2 \in X(15)$, then $x_2=4.5$, $f_2(4.5)=5$, $x_1=10-4.5=5.5$, $f_1(5.5)=2.3$, $f_1(5.5)<f_2(4.5)$; hence (5.5,4.5) is a lmaxp with $F(5.5,4.5)=6$. The set of all lmaxps is

$$\hat{X}_m = \{(10,0) \ , \ (0,10) \ , \ (3,7) \ , \ (5.5,4.5)\}$$

and the global maximum point is, by Theorem 4, the point in $\hat{X}_m$ with the largest value of F(x), namely $F(0,10)=8.25$.

Before concluding the discussion of these examples, it is interesting to note how the comprehensive knowledge of all the local extrema afforded by the criteria at hand provides illuminating information about the behavior of the objective function F(x) and its variation with the resource allocation vector x. Compare this to the traditional approaches and techniques of optimization problems which seek to determine only one feature of the objective function: its absolute minimum or maximum point. By contrast, the techniques of this paper provide the means for charting the "topography" of the objective function over its entire domain. This is vividly illustrated by the foregoing examples, which also show the dramatic changes in the topography of the objective function that might result from changes in the value of the total resource L. The knowledge of all local minima and maxima may prove quite useful in selecting suboptimal, yet more desirable, solutions under certain problem formulations. To illustrate this point, note how in Figure 5(e) the three local minimum values 3.7, 4.25, 4,5 are fairly close to each other. Suppose that in a given practical situation it is highly desirable to have the total resource as evenly distributed among the x_is as possible, in which case one may opt to choose $x=(4.25,5.75)$, $F(x)=4.25$ as the preferred "optimal" minimum solution. To state the same argument quantitatively, suppose there is an extra cost or penalty of $0.5|x_1-x_2|$ associated with the choice of any solution $x=(x_1,x_2)$ such that the objective function to be minimized is now $F(x)+0.5|x_1-x_2|$. In this case the middle local minimum in Figure 5(e) becomes the new optimal solution.

5. CONCLUSIONS

This paper has presented the "definitive" solution of the REMAXMIN problem with continuous variables: The necessary and sufficient conditions for local and global minima and maxima of the objective functions $F(x)=\max_i f_i(x_i)$ and $f(x)=\min_i f_i(x_i)$ with no restrictions (other than local monomodality) placed on the functions f_i which can be specified in analytical or graphical formats. There are a number of noteworthy theoretical and practical implications of these powerful results. The necessity and sufficiency of the conditions of optimality make them theoretically equivalent to the property of optimality. This enables the reliable determination of all the global optima of the objective function in those cases where the optimal solution is not unique, and the ordering of such multiple optima according to some additional criterion of desirability or the carrying out of a second level of optimization over the domain of all global optima. The results establish the theoretical relationship between the global optima and local optima. The determination of all local maxima and minima may be practically important because it provides significant data on the variational behavior (topography) of the objective function over the feasible set. This provides the means for carrying out suboptimal trade-offs whereby a locally optimum resource allocation is deemed more desirable than a global optimum in view of some additional criteria such as the balancing of resource allocation. The relaxation of the traditional restrictions of differentiability, monotonicity, convexity, and unimodality of $f_i(x_i)$, commonly found in previous investigations of the problem, is theoretically significant because it presents a new analytical approach for dealing with the mathematical difficulties arising from the absence of such restrictions in this problem as well as the wider contexts of optimization and nonlinear programming. The analytical concepts and techniques based on the classification of variational modes presented in this paper are totally new, as far as the author's literature search has determined. From the practical standpoint, the relaxation of the restrictions on $f_i(x_i)$ extends the applicability of results to a much wider class of real-world problems, notably in the area of multiprocessor and distributed system performance. The applicability is further enhanced by the graphical implementability of criteria and the resulting admissibility of functions $f_i(x_i)$ that are specified by their graphs or tabulated data. The only condition retained by the analysis on $f_i(x_i)$, namely local monomodality, does not seem to present any restriction of significance in practical situations.

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