Attainable Resource Allocations Bounds

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ATTAINABLE RESOURCE ALLOCATIONS BOUNDS

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Abstract

Upper and lower bounds attained by the variables of a generalized resource allocation problem are distinguished from the a priori specified bounds defining the feasible set. General theoretical criteria directly relating attainable bounds to specified bounds are presented, which are computationally superior to the traditional "modified simplex method."

Keywords: resource allocation, attainable feasible-set bounds, modified simplex method.
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Problem Formulation

Consider the resource allocation problem of mathematical programming in its most general form:

RESOURCE: optimize \( f(x_1, x_2, \ldots, x_n) \) \hspace{1cm} (1)

subject to \( \sum_{i}^{n} x_i = L \) \hspace{1cm} (2)

\( \alpha_i \leq x_i \leq \beta_i \hspace{1cm} x_i: \text{real or integer, } i = 1, 2, \ldots, n \) \hspace{1cm} (3)

where \( f \) is a real-valued function and "optimize" stands for "maximize" or "minimize" as the case may be. If \( x_i \) takes only integer values for all \( i \), RESOURCE is an integer programming problem. The set of all points \( x = (x_1, x_2, \ldots, x_n) \) satisfying the constraints in (2) and (3) is referred to as the constraint set or feasible set

\[ C(L, \alpha, \beta) = \{ x : \sum_{i}^{n} x_i = L, \hspace{1cm} x_i \in [\alpha_i, \beta_i] \} \] \hspace{1cm} (4)

where \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \). RESOURCE may now be restated as

RESOURCE: optimize \( f(x_1, x_2, \ldots, x_n) \)

subject to \( x \in C(L, \alpha, \beta) \) \hspace{1cm} (5)

We shall require the following inequality conditions to be satisfied by the specified bounds \( \alpha_i \) and \( \beta_i \):

\[ \sum_{i}^{n} \alpha_i < L < \sum_{i}^{n} \beta_i \] \hspace{1cm} (6)

For if otherwise \( \sum_{i}^{n} \alpha_i > L \) or \( \sum_{i}^{n} \beta_i > L \), there would be no points \( x \) satisfying (4), i.e., \( C(L, \alpha, \beta) \) is empty and the problem would be meaningless. Moreover, if \( \sum_{i}^{n} \alpha_i = L \) or \( \sum_{i}^{n} \beta_i = L \), the constraint set \( C(L, \alpha, \beta) \) would consist of the single point \( x = \alpha \) or \( x = \beta \) respectively, and the problem would be trivial.

This paper poses and resolves the following question: When the resource allocation vector \( x \) ranges over all points of the feasible set \( C(L, \alpha, \beta) \), what are the extreme values exhibited by each of the variables \( x_i \) and how are these extrema related
to the upper and lower bounds $\beta_i$ and $\alpha_i$ specified on $x_i$ in (3)? To state the question mathematically, define

$$a_i = \min_{x \in C(L, \alpha, \beta)} x_i \quad , \quad b_i = \max_{x \in C(L, \alpha, \beta)} x_i . \quad (7)$$

These definitions imply the following statement

$$a_i \leq x_i \leq b_i \quad \text{for all } x \in C(L, \alpha, \beta) \quad (8)$$

and the question becomes: How are the bounds $a_i$ and $b_i$ related to the specified bounds $\alpha_i$ and $\beta_i$? It might seem plausible to conjecture that $a_i = \alpha_i$ and $b_i = \beta_i$ by the reasoning that the condition $\alpha_i \leq x_i \leq \beta_i$ enters into the definition of $C(L, \alpha, \beta)$, and therefore when $x \in C$, the variable $x_i$ attains the specified lower bound $\alpha_i$ and upper bound $\beta_i$. This is not true in general as demonstrated by the following simple counterexample with two variables

$$x_1 + x_2 = L = 10 \quad , \quad x_1 \in [\alpha_1, \beta_1] = [2, 6] \quad , \quad x_2 \in [\alpha_2, \beta_2] = [5, 9]$$

for which $a_1 = \alpha_1 = 2$, $a_2 = \alpha_2 = 5$, but $b_1 = 5 < \beta_1$ and $b_2 = 8 < \beta_2$. It should be noted that the extremum values $a_i$ and $b_i$ in (7) are actually attained by the function $x_i$ for some values $x \in C$. This is a direct consequence of well-known results from mathematical analysis: a continuous function attains its extremum values over a compact set, and the set $C$ is compact because it is the intersection of the compact set $\{x : x \in [\alpha_p, \beta_p]\}$ and the closed set $\{x : \sum^n_{i=1} x_i = L\}$. The attainability of $a_i$ and $b_i$ means that there exists at least one feasible point $x \in C$ for which $x_i = a_i$ and another for which $x_i = b_i$. This might not be true for $\alpha_i$ and $\beta_i$, in which case they are said to represent unattainable bounds on $x_i$, as illustrated by $\beta_1$ and $\beta_2$ in the above example. Evidently, $a_i$ and $b_i$ cannot exceed $\alpha_i$ and $\beta_i$

$$a_i \leq \alpha_i \quad , \quad b_i \leq \beta_i \quad , \quad [a_i, b_i] \subseteq [\alpha_i, \beta_i] \quad \text{for all } i \quad (9)$$

In the remainder of this paper, we shall refer to $\alpha_i$ and $\beta_i$ as the specified bounds and to $a_i$ and $b_i$ as the attainable bounds of the resource allocation variables $x_p$, and we
shall present theoretical results that relate $a_i$ and $b_i$ to $\alpha_i$ and $\beta_i$ in a general fashion, which enables direct and efficient determination of the attainable bounds.

A significant relationship between $a_i, b_i$ and $\alpha_i, \beta_i$ is that the constraint set $C$ remains unchanged if the specified bounds are replaced by the attainable bounds, i.e.,

$$C(L,a,b) = C(L,\alpha,\beta)$$  \hspace{1cm} (10)

To prove (10), note that (9) implies $C(L,a,b) \subseteq C(L,\alpha,\beta)$ and (8) implies $C(L,\alpha,\beta) \subseteq C(L,a,b)$; hence the two sets are identical. This being the case, the solution of the general resource allocation problem described by (1)-(5) is unaffected when the attainable bounds replace the specified bounds. This raises the motivational question: Why should we be interested in determining the attainable bounds and, given $\alpha_i$ and $\beta_i$, is there something extra to be gained from an a priori knowledge of the values of $a_i$ and $b_i$? Apart from knowledge for its own sake, the following are some pragmatic reasons for the need to determine the attainable bounds:

1. Some recent theoretical results on the solution of the resource allocation problem provide criteria that are explicitly formulated in terms of, and require a priori knowledge of, the attainable bounds (see, for example, Haddad [2]).

2. Enumerative methods for solving the integer RESOURCE problem can make use of the attainable bounds to improve the efficiency of their algorithms by avoiding all points $x$ for which any $x_i \in [a_i, a_j]$ or $x_i \in [b_i, b_j]$. Such points are automatically eliminated as infeasible without checking for the condition $\Sigma_{i=1}^n x_i = L$.

3. The determination of $a_i$ and $b_i$ as stated in (7) represents an important mathematical programming problem in its own right which may arise independently from its relation to RESOURCE. The optimization problem in (7) is an instance of linear programming with bounded variables for which general solutions exist in the form of modified simplex algorithms (see, for example, Garfinkel and Nemhauser [1] and Leunberger [4]).
4. There is a fundamental logical difference between the roles that \( \alpha_i, \beta_i \) and \( a_i, b_i \) could play in any general criterion on the solution of RESOURCE. A criterion that incorporates the conditions \( x_i = \alpha_i, x_i = \beta_i \) among its requirements of optimality must, of necessity, be only a **sufficient** condition, since in general there is no guarantee that \( \alpha_i \) and \( \beta_i \) are attainable. On the other hand, a criterion that incorporates the conditions \( x_i = a_i, x_i = b_i \) can be a **necessary and sufficient** condition, since \( a_i \) and \( b_i \) are attainable. Compare as an example, the Theorem in Ibaraki and Katoh [3] which uses \( x_i = \alpha_i, x_i = \beta_i \), and is a sufficient condition, to the Theorem in [2] which uses \( x_i = a_i, x_i = b_i \) and is a necessary and sufficient condition.

In this paper we present theoretical results that relate the attainable bounds \( a_i, b_i \) to the specified bounds \( \alpha_i, \beta_i \) in a straightforward manner. The relationships between these bounds is investigated through a direct analysis, without the intermediary of the simplex method theory or its variations. Note that the computation of \( a_i \) and \( b_i \) in (7) by the existing methods would require the running of a modified simplex algorithm 2n times. The results of this paper provide the means for obtaining \( a_i \) and \( b_i \) with far less computational effort.

**Main Results**

We now present a Theorem and five corollaries that embody the main relationships between the attainable and specified bounds and also provide criteria aimed at reducing the computational effort for determining the attainable bounds. All relationships are expressed in terms of three parameters \( A, B, \) and \( d_i \) derived from the given parameters \( L, \alpha_i, \) and \( \beta_i \) as follows:

\[
A = L - \Sigma_i^a \alpha_i > 0 \quad , \quad B = \Sigma_i^a \beta_i - L > 0 \quad , \quad d_i = \beta_i - \alpha_i \geq 0 .
\]  

(11)

The positiveness of \( A \) and \( B \) and the nonnegativeness of \( d_i \) follow from (6) and (3) respectively.
Theorem

(1) If \( d_i \leq A \), then \( b_i = \beta_i \).

(2) If \( d_i \geq A \), then \( b_i = \alpha_i + A \), \( a_j = \alpha_j \) for all \( j \neq i \).

(3) If \( d_i \leq B \), then \( a_i = \alpha_i \).

(4) If \( d_i \geq B \), then \( a_i = \beta_i - B \), \( b_j = \beta_j \) for all \( j \neq i \).

Proof

(1) If, for a given value of \( i \), \( d_i = \beta_i - \alpha_i \leq A \), consider the point \( x^i \)

\[ x^i = (\alpha_1 + \theta d_1, \alpha_2 + \theta d_2, \ldots, \beta_i, \ldots, \alpha_n + \theta d_n) \]

where

\[ \theta = (A - d_i) / (A + B - d_i) \]

We now show that \( x^i \in \mathbb{C}(L, \alpha, \beta) \). Note that \( 0 \leq \theta \leq 1 \) and therefore \( \alpha_j - \alpha_i + \theta d_j \leq \beta_j \) and

\[ \Sigma_{j=1}^n x^i = \beta_i + \Sigma_{j \neq i} x^i = \beta_i + \Sigma_{j \neq i} \alpha_j + \theta \Sigma_{j \neq i} d_j \]

\[ = \beta_i + \Sigma_j \alpha_j - \alpha_i + \theta (A + B - d_i) = d_i + L - A + A - d_i = L \]

Hence \( x^i \in \mathbb{C}(L, \alpha, \beta) \), and \( \beta_i \) is attainable, i.e., \( b_i = \beta_i \).

(2) If \( d_i = \beta_i - \alpha_i \geq A \), consider the point \( x^i \)

\[ x^i = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i + A, \alpha_{i+1}, \ldots, \alpha_n) \]

Note that \( \Sigma_{j=1}^n x^i = \Sigma_{j=1}^n \alpha_j + A = L \) and \( x^i \in [\alpha_i, \beta_i] \) for all \( j \), including \( j = i \) since \( (\alpha_i + A) \leq \beta_i \). Hence \( x^i \in \mathbb{C}(L, \alpha, \beta) \) and \( \alpha_j \) is attainable for all \( j \neq i \), viz., \( a_j = \alpha_j \) for all \( j \neq i \). Furthermore, \( x^i = \alpha_i + A \) is the largest value that \( x_i \) can attain

\[ x_i = L - \Sigma_{j \neq i} x_j \leq L - \Sigma_{j \neq i} \alpha_j = \alpha_i + A \]

Hence \( b_i = \alpha_i + A \).

(3) If \( d_i = \beta_i - \alpha_i \leq B \), consider the point \( x^i \)

\[ x^i = (\beta_1 - \lambda d_1, \beta_2 - \lambda d_2, \ldots, \alpha_1, \ldots, \beta_n - \lambda d_n) \]

where
\[ \lambda = \frac{(B-d_i)}{(B + A-d_i)}. \]

Note that \(0 \leq \lambda \leq 1\); hence every \(x_j \in [\alpha_j, \beta_j]\). It is easily shown, as in part (1), that \(\Sigma x_j = L\); hence \(x^i \in C\). Thus \(\alpha_i\) is attainable and \(a_i = \alpha_i\).

(4) If \(d_i = \beta_i - \alpha_i \geq B\), consider the point \(x^i\)

\[ x^i = (\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_i - B, \beta_{i+1}, \ldots, \beta_n). \]

Note that \(\Sigma x_j = L\) and \(x_j \in [\alpha_j, \beta_j]\) for all \(j\); hence \(x^i \in C(L, \alpha, \beta)\) and \(b_j\) is attainable for all \(j \neq i\), viz., \(b_j = \beta_j\) for \(j \neq i\). Furthermore, \(x_i = \beta_i - B\) is the least value that \(x_i\) can attain.

\[ x_i = L - \Sigma_{j \neq i} x_j \geq L - \Sigma_{j \neq i} \beta_j = \beta_j - B. \]

Hence \(a_i = \beta_i - B\).

This completes the proof of the Theorem.

**Corollary 1**

\[ a_i = \max \{ \alpha_i, \beta_i - B \} \quad , \quad b_i = \min \{ \beta_i, \alpha_i + A \} \]  \hspace{1cm} (18)

**Proof:** If \(\alpha_i \geq \beta_i - B\), then \(d_i \leq B\) and from (14) one has \(a_i = \alpha_i = \max \{ \alpha_i, \beta_i - B \}\). If \(\beta_i - B \geq \alpha_i\), then \(d_i \geq B\) and from (15) one has \(a_i = \beta_i - B = \max \{ \alpha_i, \beta_i - B \}\). If \(\beta_i \leq \alpha_i + A\), then \(d_i \leq A\) and from (12) one has \(b_i = \beta_i = \min \{ \beta_i, \alpha_i + A \}\). If \(\alpha_i + A \leq \beta_i\), then \(d_i \geq A\) and from (13) one has \(b_i = \alpha_i + A = \min \{ \beta_i, \alpha_i + A \}\).

**Corollary 2**

If there is a \(d_k \geq \max \{ A, B \}\), then

(i) \(a_k = \beta_k - B\) , \(b_k = \alpha_k + A\) , \(a_j = \alpha_j\) , \(b_j = \beta_j\) for all \(j \neq k\).  \hspace{1cm} (19)

(ii) \(d_k = \max d_i\).

(iii) If in addition \(A \neq B\), then \(d_i < d_k\) for all \(i \neq k\).
Proof: Since \( d_k \geq A \) and \( d_k \geq B \), (19) follows from (13) and (15) of the Theorem. To prove part (ii), assume to the contrary that \( d_k \neq \max d_i \). Therefore there is a \( d_m > d_k > \max\{A,B\} \), and (19) is true for \( d_m \)

\[
    a_m = \beta_m - B , \quad b_m = \alpha_m + A , \quad a_j = \alpha_j , \quad b_j = \beta_j \quad \text{for all } j \neq \ell (20)
\]

Applying (19) for the specific \( j = m \) and (20) for the specific \( j = k \)

\[
    a_k = \beta_k - B , \quad b_k = \alpha_k + A , \quad a_m = \alpha_m , \quad b_m = \beta_m \quad (21)
\]

\[
    a_m = \beta_m - B , \quad b_m = \alpha_m + A , \quad a_k = \alpha_k , \quad b_k = \beta_k \quad (22)
\]

Combining (21) and (22) and noting that \( \beta_i - \alpha_i = d_i \), we obtain

\[
    d_k = d_m = A = B \quad (23)
\]

which is a contradiction to \( d_m > d_k \); hence \( d_k = \max d_i \). To prove part (iii), assume to the contrary \( d_i \neq d_k \) for all \( i \neq k \), viz., there is a \( d_m = d_k = \max\{A,B\} \). The same arguments as above can be repeated with (19), (20), (21), and (22) still valid, leading to (23), which is a contradiction to \( A \neq B \); hence \( d_i < d_k \) for all \( i \neq k \).

**Corollary 3**

If there is a \( d_k \) such that \( B \leq d_k \leq A \), then

(i) \( d_i \leq A \) for all \( i \).

(ii) If in addition \( d_i \geq B \) for all \( i \), then

\[
    a_i = \beta_i - B , \quad b_i = \beta_i \quad \text{for all } i \quad (24)
\]

**Proof:** Applying the Theorem for \( B \leq d_k \leq A \), we obtain

\[
    b_k = \beta_k , \quad a_k = \beta_k - B , \quad b_j = \beta_j \quad \text{all } j \neq k \quad (25)
\]

\[
    a_k = \beta_k - B , \quad b_j = \beta_j \quad \text{all } j \quad (26)
\]

To prove part (i), assume to the contrary that there is a \( d_m > A > B \), for which the Theorem gives

\[
    a_m = \beta_m - B , \quad b_m = \alpha_m + A , \quad a_j = \alpha_j , \quad b_j = \beta_j \quad \text{all } j \neq m \quad (27)
\]

Applying (26) for the specific \( j = m \) and (27) for the specific \( j = k \),
\[ a_k = \beta_k - B , \quad b_m = \beta_m \]  
\[ a_m = \beta_m - B , \quad b_m = \alpha_m + A , \quad a_k = \alpha_k , \quad b_k = \beta_k. \]  

Combining (28) and (29) we obtain \( d_m = \beta_m - \alpha_m = A \), which is a contradiction to \( d_m > A \); hence \( d_m \leq A \) for all \( i \). The proof of part (ii) follows immediately from (12) and (15) of the Theorem, since now we have \( B \leq d_i \leq A \) for all \( i \).

**Corollary 4**

If there is a \( d_k \) such that \( A \leq d_k \leq B \), then

(i) \( d_i \leq B \) for all \( i \).

(ii) If in addition \( d_i \geq B \) for all \( i \), then

\[ a_i = \alpha_i , \quad b_i = \alpha_i + A \quad \text{for all} \quad i. \]  

**Proof:** The proof is closely analogous to the proof of corollary 3 with the roles of \( A \) and \( B \) reversed.

**Corollary 5**

If \( d_i \leq \min\{A, B\} \) for all \( i \), then

\[ a_i = \alpha_i , \quad b_i = \beta_i \quad \text{for all} \quad i. \]  

**Proof:** Since \( d_i \leq A \) and \( d_i \leq B \), the result in (31) is a direct consequence of (12) and (14) of the Theorem.

**Examples**

Four numerical examples are compactly summarized in the table below. Rows 1, 2, and 4 show the given values of \( L, \alpha_i \), and \( \beta_i \), respectively. Rows 3 and 5 give the values of \( A \) and \( B \) as computed from (11). Row 6 gives \( d_i = \beta_i - \alpha_i \). Rows 7 and 8 give the values of the attainable bounds \( a_i \) and \( b_i \) as computed in a straightforward manner from (18) in Corollary 1. Rows 9-12 show how these bounds can be alternatively computed with less effort using Corollaries 2-5, whose conditions and expressions for \( a_i \) and \( b_i \) are restated in rows 10-12 for the convenience of the reader.
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References


