Attainable Resource Allocations Bounds

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Abstract

Upper and lower bounds attained by the variables of a generalized resource allocation problem are distinguished from the *a priori* specified bounds defining the feasible set. General theoretical criteria directly relating attainable bounds to specified bounds are presented, which are computationally superior to the traditional "modified simplex method."

Keywords: resource allocation, attainable feasible-set bounds, modified simplex method.

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Problem Formulation

Consider the resource allocation problem of mathematical programming in its most general form:

RESOURCE: optimize $f(x_1,x_2,\ldots,x_n)$ (1)

subject to $\Sigma_1^n x_i = L$ (2)

$\alpha_i \leq x_i \leq \beta_i$ x_i: real or integer, $i=1,2,\ldots,n$ (3)

where f is a real-valued function and "optimize" stands for "maximize" or "minimize" as the case may be. If x_i takes only integer values for all i, RESOURCE is an integer programming problem. The set of all points $x=(x_1,x_2,\ldots,x_n)$ satisfying the constraints in (2) and (3) is referred to as the constraint set or feasible set

$$C(L,\alpha,\beta) \equiv \{x: \Sigma_1^n x_i = L \quad , \quad x_i \in [\alpha_i,\beta_i]\} \tag{4}$$

where $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\ldots,\beta_n)$. RESOURCE may now be restated as

RESOURCE: optimize $f(x_1,x_2,\ldots,x_n)$

subject to $x \in C(L,\alpha,\beta)$ (5)

We shall require the following inequality conditions to be satisfied by the specified bounds α_i and β_i:

$$\Sigma_1^n \alpha_i < L < \Sigma_1^n \beta_i \; . \tag{6}$$

For if otherwise $\Sigma_1^n \alpha_i > L$ or $\Sigma_1^n \beta_i > L$, there would be no points x satisfying (4), i.e., $C(L,\alpha,\beta)$ is empty and the problem would be meaningless. Moreover, if $\Sigma_1^n \alpha_i = L$ or $\Sigma_1^n \beta_i = L$, the constraint set $C(L,\alpha,\beta)$ would consist of the single point $x=\alpha$ or $x=\beta$ respectively, and the problem would be trivial.

This paper poses and resolves the following question: When the resource allocation vector x ranges over all points of the feasible set $C(L,\alpha,\beta)$, what are the extreme values exhibited by each of the variables x_i and how are these extrema related to the upper and lower bounds β_i and α_i specified on x_i in (3)? To state the question mathematically, define

$$a_i \equiv \min_{x \in C(L,\alpha,\beta)} x_i \quad , \quad b_i \equiv \max_{x \in C(L,\alpha,\beta)} x_i \; . \tag{7}$$

These definitions imply the following statement

$$a_i \leq x_i \leq b_i \qquad \text{for all } x \in C(L,\alpha,\beta) \tag{8}$$

and the question becomes: How are the bounds a_i and b_i related to the specified bounds α_i and β_i? It might seem plausible to conjecture that $a_i = \alpha_i$ and $b_i = \beta_i$ by the reasoning that the condition $\alpha_i \leq x_i \leq \beta_i$ enters into the definition of $C(L,\alpha,\beta)$, and therefore when $x \in C$, the variable x_i attains the specified lower bound α_i and upper bound β_i. This is not true in general as demonstrated by the following simple counterexample with two variables

$$x_1 + x_2 = L = 10 \quad , \quad x_1 \in [\alpha_1, \beta_1] = [2,6] \quad , \quad x_2 \in [\alpha_2, \beta_2] = [5,9]$$

for which $a_1 = \alpha_1 = 2$, $a_2 = \alpha_2 = 5$, but $b_1 = 5 < \beta_1$ and $b_2 = 8 < \beta_2$. It should be noted that the extremum values a_i and b_i in (7) are actually *attained* by the function x_i for some values $x \in C$. This is a direct consequence of well-known results from mathematical analysis: a continuous function attains its extremum values over a compact set, and the set C is compact because it is the intersection of the compact set $\{x : x_i \in [\alpha_i, \beta_i]\}$ and the closed set $\{x : \Sigma_1^n x_i = L\}$. The attainability of a_i and b_i means that there exists at least one feasible point $x \in C$ for which $x_i = a_i$ and another for which $x_i = b_i$. This might not be true for α_i and β_i, in which case they are said to represent unattainable bounds on x_i, as illustrated by β_1 and β_2 in the above example. Evidently, a_i and b_i cannot exceed α_i and β_i

$$a_i \leq \alpha_i \quad , \quad b_i \leq \beta_i \quad , \quad [a_i, b_i] \subseteq [\alpha_i, \beta_i] \qquad \text{for all } i \tag{9}$$

In the remainder of this paper, we shall refer to α_i and β_i as the *specified bounds* and to a_i and b_i as the *attainable bounds* of the resource allocation variables x_i, and we shall present theoretical results that relate a_i and b_i to α_i and β_i in a general fashion, which enables direct and efficient determination of the attainable bounds.

A significant relationship between a_i, b_i and α_i, β_i is that the constraint set C remains unchanged if the specified bounds are replaced by the attainable bounds, i.e.,

$$C(L,a,b) = C(L,\alpha,\beta) \tag{10}$$

To prove (10), note that (9) implies $C(L,a,b) \subseteq C(L,\alpha,\beta)$ and (8) implies $C(L,\alpha,\beta) \subseteq C(L,a,b)$; hence the two sets are identical. This being the case, the solution of the general resource allocation problem described by (1)-(5) is unaffected when the attainable bounds replace the specified bounds. This raises the motivational question: Why should we be interested in determining the attainable bounds and, given α_i and β_i, is there something extra to be gained from an *a priori* knowledge of the values of a_i and b_i? Apart from knowledge for its own sake, the following are some pragmatic reasons for the need to determine the attainable bounds:

1. Some recent theoretical results on the solution of the resource allocation problem provide criteria that are explicitly formulated in terms of, and require *a priori* knowledge of, the attainable bounds (see, for example, Haddad [2]).
2. Enumerative methods for solving the integer RESOURCE problem can make use of the attainable bounds to improve the efficiency of their algorithms by avoiding all points x for which any $x_i \in [\alpha_i, a_i]$ or $x_i \in [b_i, \beta_i]$. Such points are automatically eliminated as infeasible without checking for the condition $\Sigma_1^n x_i = L$.
3. The determination of a_i and b_i as stated in (7) represents an important mathematical programming problem in its own right which may arise independently from its relation to RESOURCE. The optimization problem in (7) is an instance of linear programming with bounded variables for which general solutions exist in the form of modified simplex algorithms (see, for example, Garfinkel and Nemhauser [1] and Leunberger [4]).

4. There is a fundamental logical difference between the roles that α_i,β_i and a_i,b_i could play in any general criterion on the solution of RESOURCE. A criterion that incorporates the conditions $x_i=\alpha_i$, $x_i=\beta_i$ among its requirements of optimality must, of necessity, be only a *sufficient* condition, since in general there is no guarantee that α_i and β_i are attainable. On the other hand, a criterion that incorporates the conditions $x_i=a_i$, $x_i=b_i$ can be a *necessary and sufficient* condition, since a_i and b_i are attainable. Compare as an example, the Theorem in Ibaraki and Katoh [3] which uses $x_i=\alpha_i$, $x_i=\beta_i$, and is a sufficient condition, to the Theorem in [2] which uses $x_i=a_i$, $x_i=b_i$ and is a necessary and sufficient condition.

In this paper we present theoretical results that relate the attainable bounds a_i,b_i to the specified bounds α_i,β_i in a straightforward manner. The relationships between these bounds is investigated through a direct analysis, without the intermediary of the simplex method theory or its variations. Note that the computation of a_i and b_i in (7) by the existing methods would require the running of a modified simplex algorithm 2n times. The results of this paper provide the means for obtaining a_i and b_i with far less computational effort.

Main Results

We now present a Theorem and five corollaries that embody the main relationships between the attainable and specified bounds and also provide criteria aimed at reducing the computational effort for determining the attainable bounds. All relationships are expressed in terms of three parameters A, B, and d_i derived from the given parameters L, α_i, and β_i as follows:

$$A \equiv L-\Sigma_1^n\alpha_i > 0 \quad , \quad B \equiv \Sigma_1^n\beta_i-L > 0 \quad , \quad d_i \equiv \beta_i-\alpha_i \geq 0 \; . \tag{11}$$

The positiveness of A and B and the nonnegativeness of d_i follow from (6) and (3) respectively.

Theorem

(1) If $d_i \leq A$, then $b_i = \beta_i$. (12)

(2) If $d_i \geq A$, then $b_i = \alpha_i + A$, $a_j = \alpha_j$ for all $j \neq i$. (13)

(3) If $d_i \leq B$, then $a_i = \alpha_i$. (14)

(4) If $d_i \geq B$, then $a_i = \beta_i - B$, $b_j = \beta_j$ for all $j \neq i$. (15)

Proof

(1) If, for a given value of i, $d_i = \beta_i - \alpha_i \leq A$, consider the point x^i

$$x^i = (\alpha_1 + \theta d_1 , \alpha_2 + \theta d_2 , \ldots , \beta_i , \ldots , \alpha_n + \theta d_n) \tag{16}$$

where

$$\theta \equiv (A - d_i)/(A + B - d_i) . \tag{17}$$

We now show that $x^i \in C(L,\alpha,\beta)$. Note that $0 \leq \theta \leq 1$ and therefore $\alpha_j \leq \alpha_j + \theta d_j \leq \beta_j$ and

$$\Sigma^n_{j=1} x^i_j = \beta_i + \Sigma_{j \neq i} x^i_j = \beta_i + \Sigma_{j \neq i}\alpha_j + \theta \Sigma_{j \neq i} d_j$$

$$= \beta_i + \Sigma_j \alpha_j - \alpha_i + \theta(A + B - d_i) = d_i + L - A + A - d_i = L .$$

Hence $x^i \in C(L,\alpha,\beta)$, and β_i is attainable, i.e., $b_i = \beta_i$.

(2) If $d_i = \beta_i - \alpha_i \geq A$, consider the point x^i

$$x^i = (\alpha_1 , \alpha_2 , \ldots , \alpha_{i-1} , \alpha_i + A , \alpha_{i+1} , \ldots , \alpha_n) .$$

Note that $\Sigma^n_{j=1} x^i_j = \Sigma^n_{j=1}\alpha_j + A = L$ and $x^i_j \in [\alpha_j, \beta_j]$ for all j, including $j = i$ since $(\alpha_i + A) \leq \beta_i$. Hence $x^i \in C(L,\alpha,\beta)$ and α_j is attainable for all $j \neq i$, viz., $a_j = \alpha_j$ for all $j \neq i$. Furthermore, $x^i_i = \alpha_i + A$ is the largest value that x_i can attain

$$x_i = L - \Sigma_{j \neq i} x_j \leq L - \Sigma_{j \neq i}\alpha_j = \alpha_i + A .$$

Hence $b_i = \alpha_i + A$.

(3) If $d_i = \beta_i - \alpha_i \leq B$, consider the point x^i

$$x^i = (\beta_1 - \lambda d_1 , \beta_2 - \lambda d_2 , \ldots , \alpha_i , \ldots , \beta_n - \lambda d_n)$$

where

$$\lambda \equiv (B-d_i)/(B + A-d_i) .$$

Note that $0 \leq \lambda \leq 1$; hence every $x_j^i \in [\alpha_j, \beta_j]$. It is easily shown, as in part (1), that $\Sigma x_j^i = L$; hence $x^i \in C$. Thus α_i is attainable and $a_i = \alpha_i$.

(4) If $d_i = \beta_i - \alpha_i \geq B$, consider the point x^i

$$x^i = (\beta_1 , \beta_2 , \dots , \beta_{i-1} , \beta_i - B , \beta_{i+1} , \dots , \beta_n) .$$

Note that $\Sigma x_j^i = L$ and $x_j^i \in [\alpha_j, \beta_j]$ for all j; hence $x^i \in C(L,\alpha,\beta)$ and b_j is attainable for all $j \neq i$, viz., $b_j = \beta_j$ for $j \neq i$. Furthermore, $x_i^i = \beta_i - B$ is the least value that x_i can attain

$$x_i = L - \Sigma_{j \neq i} x_j \geq L - \Sigma_{j \neq i} \beta_j = \beta_j - B.$$

Hence $a_i = \beta_i - B$.

This completes the proof of the Theorem.

Corollary 1

$$a_i = \max \{\alpha_i , \beta_i - B\} \quad , \quad b_i = \min \{\beta_i , \alpha_i + A\} \tag{18}$$

Proof: If $\alpha_i \geq \beta_i - B$, then $d_i \leq B$ and from (14) one has $a_i = \alpha_i = \max\{\alpha_i, \beta_i - B\}$. If $\beta_i - B \geq \alpha_i$, then $d_i \geq B$ and from (15) one has $a_i = \beta_i - B = \max\{\alpha_i, \beta_i - B)$. If $\beta_i \leq \alpha_i + A$, then $d_i \leq A$ and from (12) one has $b_i = \beta_i = \min\{\beta_i, \alpha_i + A\}$. If $\alpha_i + A \leq \beta_i$, then $d_i \geq A$ and from (13) one has $b_i = \alpha_i + A = \min\{\beta_i, \alpha_i + A\}$.

Corollary 2

If there is a $d_k \geq \max\{A,B\}$, then

(i) $a_k = \beta_k - B \quad , \quad b_k = \alpha_k + A \quad , \quad a_j = \alpha_j \quad , \quad b_j = \beta_j \quad$ for all $j \neq k$. (19)

(ii) $d_k = \max d_i$.

(iii) If in addition $A \neq B$, then $d_i < d_k$ for all $i \neq k$.

Proof: Since $d_k \geq A$ and $d_k \geq B$, (19) follows from (13) and (15) of the Theorem. To prove part (ii), assume to the contrary that $d_k \neq \max d_i$. Therefore there is a $d_m > d_k \geq \max\{A,B\}$, and (19) is true for d_m

$$a_m = \beta_m - B \quad , \quad b_m = \alpha_m + A \quad , \quad a_j = \alpha_j \quad , \quad b_j = \beta_j \qquad \text{for all } j \neq m \tag{20}$$

Applying (19) for the specific $j=m$ and (20) for the specific $j=k$

$$a_k = \beta_k - B \quad , \quad b_k = \alpha_k + A \quad , \quad a_m = \alpha_m \quad , \quad b_m = \beta_m \tag{21}$$

$$a_m = \beta_m - B \quad , \quad b_m = \alpha_m + A \quad , \quad a_k = \alpha_k \quad , \quad b_k = \beta_k \, . \tag{22}$$

Combining (21) and (22) and noting that $\beta_i - \alpha_i = d_i$, we obtain

$$d_k = d_m = A = B \tag{23}$$

which is a contradiction to $d_m > d_k$; hence $d_k = \max d_i$. To prove part (iii), assume to the contrary $d_i \nless d_k$ for all $i \neq k$, viz., there is a $d_m = d_k = \max\{A,B\}$. The same arguments as above can be repeated with (19), (20), (21), and (22) still valid, leading to (23), which is a contradiction to $A \neq B$; hence $d_i < d_k$ for all $i \neq k$.

Corollary 3

If there is a d_k such that $B \leq d_k \leq A$, then

(i) $d_i \leq A$ for all i.

(ii) If in addition $d_i \geq B$ for all i, then

$$a_i = \beta_i - B \quad , \quad b_i = \beta_i \quad \text{for all } i. \tag{24}$$

Proof: Applying the Theorem for $B \leq d_k \leq A$, we obtain

$$b_k = \beta_k \quad , \quad a_k = \beta_k - B \quad , \quad b_j = \beta_j \qquad \text{all } j \neq k \tag{25}$$

$$a_k = \beta_k - B \quad , \quad b_j = \beta_j \qquad \text{all } j \, . \tag{26}$$

To prove part (i), assume to the contrary that there is a $d_m > A \geq B$, for which the Theorem gives

$$a_m = \beta_m - B \quad , \quad b_m = \alpha_m + A \quad , \quad a_j = \alpha_j \quad , \quad b_j = \beta_j \qquad \text{all } j \neq m \tag{27}$$

Applying (26) for the specific $j=m$ and (27) for the specific $j=k$,

$$a_k = \beta_k - B \quad , \quad b_m = \beta_m \tag{28}$$

$$a_m = \beta_m - B \quad , \quad b_m = \alpha_m + A \quad , \quad a_k = \alpha_k \quad , \quad b_k = \beta_k \,. \tag{29}$$

Combining (28) and (29) we obtain $d_m = \beta_m - \alpha_m = A$, which is a contradiction to $d_m > A$; hence $d_m \leq A$ for all i. The proof of part (ii) follows immediately from (12) and (15) of the Theorem, since now we have $B \leq d_i \leq A$ for all i.

Corollary 4

If there is a d_k such that $A \leq d_k \leq B$, then

(i) $d_i \leq B$ for all i.

(ii) If in addition $d_i \geq B$ for all i, then

$$a_i = \alpha_i \quad , \quad b_i = \alpha_i + A \qquad \text{for all } i \,. \tag{30}$$

Proof: The proof is closely analogous to the proof of corollary 3 with the roles of A and B reversed.

Corollary 5

If $d_i \leq \min\{A,B\}$ for all i, then

$$a_i = \alpha_i \quad , \quad b_i = \beta_i \qquad \text{for all } i \,. \tag{31}$$

Proof: Since $d_i \leq A$ and $d_i \leq B$, the result in (31) is a direct consequence of (12) and (14) of the Theorem.

Examples

Four numerical examples are compactly summarized in the table below. Rows 1, 2, and 4 show the given values of L, α_i, and β_i, respectively. Rows 3 and 5 give the values of A and B as computed from (11). Row 6 gives $d_i = \beta_i - \alpha_i$. Rows 7 and 8 give the values of the attainable bounds a_i and b_i as computed in a straightforward manner from (18) in Corollary 1. Rows 9-12 show how these bounds can be alternatively computed with less effort using Corollaries 2-5, whose conditions and expressions for a_i and b_i are restated in rows 10-12 for the convenience of the reader.

		Example 1			Example 2			Example 3			Example 4		
1	L	15			16			14			20		
2	α_i	4	2	3	3	4	5	2	0	1	2	3	4
3	A	6			4			11			11		
4	β_i	5	9	5	8	8	10	6	3	3	10	11	9
5	B	4			10			1			10		
6	d_i	1	7	2	5	4	5	4	3	3	8	9	5
7	a_i	4	5	3	3	4	5	5	2	2	2	3	4
8	b_i	5	8	5	7	8	9	6	3	3	10	11	9
9	Criterion	Corollary 2			Corollary 4			Corollary 3			Corollary 5		
10	Conditions	$d_2 \geq \max\{A,B\}$			$A \leq d_i \leq B$			$B \leq d_i \leq A$			$d_i \leq \min\{A,B\}$		
11	a_i	$a_2 = \beta_2 - B, a_i = \alpha_i$			$a_i = \alpha_i$			$a_i = \beta_i - B$			$a_i = \alpha_i$		
12	b_i	$b_2 = \alpha_2 + A, b_i = \beta_i$			$b_i = \alpha_i + A$			$b_i = \beta_i$			$b_i = \beta_i$		

References

[1] R. S. Garfinkel and G. L. Nemhauser, Integer Programming, John Wiley & Sons, pp. 43-48, 1972.

[2] E. K. Haddad, "Partitioned Load Allocation for Minimum Parallel Processing Time," Proceedings of the 1989 International Conference on Parallel Processing, August 1989.

[3] T. Ibaraki and N. Katoh, Resource Allocation Problems: Algorithmic Approaches, MIT Press, p. 31, 1988.

[4] D. G. Luenberger, Introduction to Linear and Nonlinear Programming, Addison-Wesley, pp. 48-53, 1973.