A Homotopy Approach for Solving Constrained Optimization Problems

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Abstract

A homotopy approach for solving constrained parameter optimization problems is examined. The first order necessary conditions, with the complementarity conditions represented using a technique due to Mangasarian, are solved. The equations are augmented to avoid singularities which occur when the active constraint changes. The Chow-Yorke algorithm is used to track the homotopy path leading to the solution to the desired problem at the terminal point. A simple example which illustrates the technique, and an application to a fuel optimal orbital transfer problem are presented.

Introduction

In solving constrained parameter optimization problems, a known solution to some problem is often used as an initial estimate for solving a similar problem with different constants. The reasoning behind this being that if there are small variations in the problem constants, the problem characteristics are essentially unchanged, and the solution to the new problem should be in the neighborhood of the initial guess. Unfortunately, as many would attest, this procedure often fails to provide a solution.

In spite of these failures, this method of obtaining solutions by systematically varying the problem constants is very appealing. The procedure of varying the system constants either in a discrete manner or in a continuous manner has been successfully applied to a number of problems. Gfrerer, Guddat and Wacker[1] proposed an algorithm based on this continuation idea coupled with an active constraint set strategy to solve constrained optimization problems. The algorithm starts with the solution to some known problem, with an index set keeping track of the active constraints. An estimate of the solution at the next parameter point is obtained based on the Jacobian matrix at the current point. This estimate is then used in a corrector iteration. The inequality constraints and their associated Lagrange multipliers are monitored. If at any step the status of an

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inequality constraint changes, the precise point at which it changes is found. The index set is then updated and the method proceeds as before until the desired final values of the parameters are attained.

This method, however, fails when the Jacobian matrix becomes singular, which can occur if the parameters are not monotonic with respect to the continuation parameter or if they are multivalued. Under these conditions, this method stops. This situation, however, can be tackled using Keller's algorithm[2] for handling bifurcation problems, giving us different procedures for different situations. Also, there exist situations where if a hitherto active constraint is removed from the constraint set when the associated Lagrange multiplier becomes zero, the solution becomes vastly different than if the constraint were left in with a zero multiplier. Thus an appropriate active constraint strategy is required.

Our effort stems from attempting to devise a procedure which is able to handle inequality constraints with ease, and at the same time uniformly deal with problems where the Jacobian matrix becomes singular. Our approach is based on using the Chow-Yorke algorithm[3] to solve the Fritz-John[4] equations. The complementarity conditions on the inequality constraints are represented using a technique of Mangasarian[5].

We have used this algorithm to solve several problems, including a fuel-optimal orbital rendezvous.

First Order Necessary Conditions for Optimality

Consider a constrained optimization problem: Find $\bar{x}$ such that

$$C(\bar{x}) = \underset{x \in X}{\mathrm{Min}}\, C(x)\,; \quad \bar{x} \in X = \{ x \mid x \in \mathscr{R}^n,\ g(x) \geq 0,\ \ h(x) = 0 \} \qquad [1]$$

where $C : \mathscr{R}^n \rightarrow \mathscr{R}$, $g : \mathscr{R}^n \rightarrow \mathscr{R}^{ni}$ and $h : \mathscr{R}^n \rightarrow \mathscr{R}^{ne}$ are twice continuously differentiable functions. The generalized Kuhn-Tucker conditions or equivalently the Fritz-John conditions[4] are:

If $\overline{x}$ is a solution to [1] then there exists a scalar $\overline{\mu}_0$ and vectors $\overline{\lambda} \in \mathscr{R}^{ne}$ and $\overline{\mu} \in \mathscr{R}^{ni}$ such that:

$$\overline{\mu}_0 \; \nabla \, C(\overline{x}) \; - \; \overline{\lambda} \; \nabla \, h(\overline{x}) \; - \; \overline{\mu} \; \nabla \, g(\overline{x}) = 0, \qquad [2]$$

$$h(\overline{x}) = 0, \qquad [3]$$

$$\overline{\mu} \; g(\overline{x}) = 0, \qquad [4]$$

$$g(\overline{x}) \; \geq \; 0 \,, \quad \overline{\mu} \; \geq \; 0, \qquad [5]$$

$$\overline{\mu}_0 \geq 0 \,, \quad [\overline{\lambda}, \overline{\mu}, \overline{\mu}_0] \neq 0. \qquad [6]$$

We normally assume $\overline{\mu}_0 = 1$ (the Kuhn-Tucker conditions), hoping that the Kuhn-Tucker constraint qualification[4] will be satisfied at $\overline{x}$.

Equivalent Non-Linear Algebraic Equations

Often, we are interested in obtaining solutions to some problems with small variations in the system parameters. These parameters may occur in the cost function, in the equality constraints, in the inequality constraints, or in all of them. In principle we would start with the solution to a known problem and then in some systematic manner solve a sequence of problems varying these system parameters until the desired solution is attained. We could scale these parameters in such a way that we have to monitor only one parameter σ, for example,

$$c = \sigma \;\; c_2 + (1 - \sigma) \, c_1,$$

where c_1 is the parameter vector for the known problem and c_2 is the final desired value.

We could do this tracking by solving the equality expressions of the Kuhn-Tucker conditions and successively varying this parameter σ. This procedure will indeed provide a solution if the second order necessary conditions are satisfied with strict complementary slackness, i.e., for each index i, $\mu_i > 0$ or $g_i(x, \sigma) > 0$, and if the gradients of the active constraints remain linearly independent[6]. Thus, if the variation of the parameter causes the active constraint set to change, one is liable to obtain erroneous results. Hence, we need some way of representing the conditions on the inequality constraints [4], [5] in the form of an equality to allow us to use curve tracking schemes to solve these as a set of nonlinear algebraic equations. Note that conditions [4], [5] are in the complementarity form. This enables us to make use of a result due to Mangasarian[5].

Mangasarian's Complementarity Theorem

Let $\Theta : \mathscr{R} \rightarrow \mathscr{R}$ *be any strictly increasing function with* $\Theta(0) = 0$.
Then $z \in \mathscr{R}^n$ *and* $f \in \mathscr{R}^n$ *solve the complementarity conditions*

$$z \geq 0, \quad f \geq 0 \quad \text{and} \quad z^* f = 0$$

if and only if z *and* f *satisfy*

$$\Theta(| f_i - z_i |) - \Theta(f_i) - \Theta(z_i) = 0 \qquad [7]$$

for $i = 1, \ldots, n$.

Thus the Kuhn-Tucker conditions for the parameter dependent problem

$$\min_{x \in X} C(x, \sigma)\,; \quad X = \{ x \mid x \in \mathscr{R}^n, \ g(x, \sigma) \geq 0\,,\ h(x, \sigma) = 0 \} \qquad [1a]$$

can be written as

$$\nabla C(x, \sigma) - \lambda \nabla h(x, \sigma) - \mu \nabla g(x, \sigma) = 0, \qquad [2a]$$

$$h(x, \sigma) = 0, \qquad [3a]$$

$$-\Theta(|\ g_i(x, \sigma) - \mu_i\ |) + \Theta(g_i(x, \sigma)) + \Theta(\mu_i) = 0, \quad i = 1, \ldots, ni. \qquad [8]$$

It is assumed that $\Theta(|\ t\ |)$ is at least C^2. It is now a simple matter to use a curve tracking algorithm to connect the solution to the problem at $\sigma = 0$ to the solution of the problem of interest at $\sigma = 1$. However, as is immediately transparent on examining the Jacobian matrix of the above system of non-linear equations with respect to $\{ x, \lambda, \mu \}$, the Jacobian matrix becomes singular if $\mu_i = g_i(x, \sigma) = 0$ for some index i, which is something we would like to avoid. Such a situation occurs when going on or off a constraint, and indeed forms the basis for the logic associated with most active constraint strategies.

Here we make use of a suggestion due to Watson[7] to avoid this singularity. We modify [8] as follows:

$$\sigma\{-\Theta(|\ g_i(x, \sigma) - \mu_i\ |) + \Theta(g_i(x, \sigma)) + \Theta(\mu_i)\} + (1-\sigma)\ (\mu_i - \bar{a}_i) = 0 \qquad [9]$$

where $\bar{a} \in \mathscr{R}^{ni}$ is chosen such that equations [8] and [9] are satisfied at $\sigma = 0$. With this modification, the path of solutions to the system of non-linear equations given by [2a], [3a] and [9] yields a candidate optimal solution only at $\sigma = 1$.

Another possibility is to use the same type of modification to [2a] and [3a], i.e.,

$$\sigma[\ \nabla C(x, \sigma) - \lambda\ \nabla h(x, \sigma) - \mu\ \nabla g(x, \sigma)\ ] + (1-\sigma)(x - \bar{b}) = 0, \qquad [2b]$$

$$\sigma[\ h(x, \sigma)\ ] + (1-\sigma)(\lambda - \bar{c}) = 0 \qquad [3b]$$

where $\bar{b} \in \mathscr{R}^n$, $\bar{c} \in \mathscr{R}^{ne}$.

The motivation to do this comes from the Chow-Yorke algorithm based on the Parametrized Sard's Theorem[3,8,9].

Definition Let $U, V \subset \mathscr{R}^n$ be open sets and $\rho : U \times [0, 1) \times V \to \mathscr{R}^n$ be a C^2 map. ρ is said to be transversal to zero if the Jacobian matrix $D\rho$ has full rank on $\rho^{-1}(0)$.

Parametrized Sard's Theorem

If $\rho(a, \sigma, z)$ *is transversal to zero, then for almost all* $a \in U$, *the map*

$$\rho_a(\sigma, z) = \rho(a, \sigma, z)$$

is also transversal to zero, i.e., with probability one the Jacobian matrix $D\rho_a(\sigma, z)$ *has full rank on* $\rho_a^{-1}(0)$.

Thus if $\rho(a, \sigma, z)$ is a homotopy between a simple map $g_a(z) = \rho(a, 0, z)$ and the map of interest $f(z) = \rho(a, 1, z)$, and if ρ is transversal to zero, then $\rho_a(\sigma, z)$ (with the vector a fixed) is also transversal to zero for almost all choices of a. The import of this is that the zero set of $\rho_a(\sigma, z)$ consists of smooth, disjoint, non-bifurcating curves, which under suitable hypothesis connect the zeroes of g_a to those of f. The Chow-Yorke algorithm is to track the zero curve of $\rho_a(\sigma, z)$ emanating from the (known) zeroes of $g_a(z)$.

The procedure then is:

1. Construct a homotopy map $\rho(a, \sigma, z)$ such that the Jacobian matrix $D\rho$ has full rank on $\rho^{-1}(0)$, $\rho(a, 0, z) = g_a(z)$, and $\rho(a, 1, z) = f(z)$.
2. Choose $g_a(z)$ with a unique known root, or show that the zero curves of ρ_a emanating from $\sigma = 0$ are monotone in σ if $\rho_a(0, z) = g_a(z) = 0$ has more than one solution.
3. Show that the zero curves of $\rho_a(\sigma, z)$ are bounded.

Then the supporting theory[3,8] guarantees that for almost all a there exists a zero curve from $\sigma = 0$ to a root $\bar{z}$ of $f(\bar{z}) = 0$ at $\sigma = 1$ and that this curve has finite arc length if $Df(\bar{z})$ is nonsingular.

Usually 3. is very difficult to show and in some cases may not even be true. So in general a curve starting from $\sigma = 0$ can either

a. reach a root $\bar{z}$ of $f(z)$ at $\sigma = 1$, or
b. wander off to infinity.

Numerical Results

The Chow-Yorke algorithm, based on [2b], [3b], and [9] was applied to the problems described below. The curve tracking was done using the code HOMPACK[10].

Example 1

$$P(\bar{x}, \sigma) = \min_{x \in X} \; x_1^2 + x_1 x_2 + (\alpha - 1.0)\, x_2^2$$

$$\bar{x} \in X = \{ x \mid x \in \mathcal{R}^2, \; g(x) \geq 0 \}$$

where the inequality constraints are defined as

$$g_1(x) \equiv 2x_2 - x_1 \geq 0$$

$$g_2(x) \equiv 2x_1 - x_2 \geq 0$$

$$g_3(x) \equiv x_1^2 + x_2^2 - 1 \geq 0$$

$$g_4(x) \equiv 2 - x_1^2 - x_2^2 \geq 0$$

and $\alpha = \sigma \alpha_1 + (1 - \sigma)\, \alpha_0$, $\alpha_0 = 0$, $\alpha_1 = 3$.

$\alpha_0 = 0$ is the starting problem for which we know the solution to be

$$x_1 = 0.6324555, \quad x_2 = 1.2649111, \ \mu_1 = 0.0, \ \mu_2 = 1.3914022, \ \mu_3 = 0.0, \ \mu_4 = 0.2 \ .$$

The feasible region for this problem is as shown in Figure 1. This is a simple example for which the solution is known analytically for different values of α. It can be easily shown that depending on the value of α each of the corners could be a locally optimal solution. For this problem for $\alpha \leq 0.25$ the top, left corner is the optimal solution. At $\alpha = 0.25$ all feasible points on the constraint g_1 are solutions. For $0.25 \leq \alpha \leq 2.75$ the bottom left corner provides a locally optimal solution, and for $1.25 \leq \alpha$ the bottom right corner provides a locally optimal solution. Note that there are multiple solutions for the values of α for $1.25 \leq \alpha \leq 2.75$.

If one were to use a simple continuation method like that of Gfrerer, Guddat and Wacker[1], starting at the solution to the problem at $\alpha = 0.0$, the marching procedure would break down at $\alpha = 0.25$ since the Jacobian matrix is singular at this value of α. So one has to resort to the homotopy curve tracking procedure with the arc length as the parameter and α as simply another dependent variable.

The Mangasarian complementarity function used for our problems is a cubic, i.e., $\Theta(t) = t^3$, since this gives us the simplest function for which $\Theta(|t|)$ is C^2. We started with the solution at $\alpha = 0$ and used the map given by [2b], [3b], and [9], assuming $\mu_0 = 1$. For an accuracy of 10^{-12} , the curve tracking procedure required 69 Jacobian evaluations, and the arc length of the connecting path was 3.297855. Figure 1 also gives the plot of x_1 versus x_2 while varying σ from $\sigma = 0$ to $\sigma = 1$. As can be seen, the constraints are not satisfied everywhere along the homotopy zero curve. Figure 2 gives the history of the variables and the Lagrange multipliers for $\sigma = 0$ to $\sigma = 1 (\alpha = 3)$.

Example 2 - Fuel-Optimal Orbital Rendezvous Problem

The problem is to find a minimum fuel rendezvous trajectory between two bodies, the non-maneuvering target and the interceptor. The interceptor trajectory consists of Keplerian coasting arcs separated by impulsive thrusting, characterized by a change in velocity (magnitude and direction). A final impulse is applied at the end of the interceptor trajectory to provide a velocity match with the target. Hence the number of impulses equals the number of coasting arcs. The maneuver must be completed within some specified time and the trajectory must avoid passing through the earth, i.e., the arcs must not violate a minimum radius constraint. The fuel-optimal problem translates to minimizing the total change in the velocity (characteristic velocity).

The variables are: the coasting angles on each arc including a possible initial coast, components of the velocity change vector, and the coasting angle of the target.

For our present analysis, we are assuming a spherical earth. To represent the trajectory equations, we use Burdet oscillator[11,12] type co-ordinates with the change in true anomaly as the independent variable. Thus, the position and velocity of the body in a Keplerian orbit can be represented by:

- u and $\hat{r}$ — reciprocal of the magnitude of the radius vector, and a unit vector in the radial direction;
- h and $\hat{h}$ — magnitude of the angular momentum vector, and a unit vector along its direction;
- $\vec{r}(.)$ — the radius vector given by $\dfrac{\hat{r}(.)}{u(.)}$;
- $\vec{v}(.)$ — the velocity vector given by $h(.)\ \{\ u(.)\,\hat{r}'(.)\ -\ u'(.)\,\hat{r}(.)\ \}$;

where $(\,)'$ refers to the derivative of $(\,)$ with respect to the change in true anomaly.

Therefore, knowing initial conditions on any subarc and the change in true anomaly, the conditions at any other point can be obtained as

$$u(\eta) = \frac{\mu}{h^2} + (u(0) - \frac{\mu}{h^2}) \cos(\eta) + u'(0) \sin(\eta), \qquad [10]$$

$$u'(\eta) = -(u(0) - \frac{\mu}{h^2}) \sin(\eta) + u'(0) \cos(\eta), \qquad [11]$$

and similarly the unit vectors as

$$\hat{r}(\eta) = \hat{r}(0) \cos(\eta) + \hat{r}'(0) \sin(\eta), \qquad [12]$$

$$\hat{r}'(\eta) = -\hat{r}(0) \sin(\eta) + \hat{r}'(0) \cos(\eta), \qquad [13]$$

$$\hat{h}(\eta) = \hat{h}(0). \qquad [14]$$

The time of flight T on any subarc can be obtained by integrating

$$T(\eta) = \int_0^{\eta} \frac{1}{h\, u^2(\theta)}\, d\theta. \qquad [15]$$

At an impulse u and $\hat{r}$ remain unchanged and the impulse is characterized by a change in u', h, $\hat{r}'$, and $\hat{h}$. Thus a change in u' and h provides the magnitude change in velocity and change in $\hat{r}'$ and $\hat{h}$ provides the directional change. Since $\hat{r}$ is fixed, the only change, if any, in $\hat{r}'$ and $\hat{h}$ is a rotation ϕ about $\hat{r}$. Using these Burdet oscillator type co-ordinates to represent the position and velocity, an impulse vector $\{\Delta V_x, \Delta V_y, \Delta V_z\}$ is characterized by $\{ \Delta u', \Delta h, \phi \}$.

Mathematically, the above problem can be described as choosing a sequence of $\{\eta, \Delta u', \Delta h, \phi\}$ so that the characteristic velocity (total velocity change), which provides a measure of the fuel consumed, is minimized[13].

Therefore, a time limited problem becomes

$$\min_{S} \quad V(x), \qquad [16]$$

where $S = \{ (\eta, \Delta u', \Delta h, \phi)_j, \eta_t \;, j = 1, \ldots, \text{nim},\}$ where nim = prespecified number of impulses, and the characteristic velocity V can be expressed in terms of these variables as:

$$V = \sum_{j=1}^{Nim} \sqrt{u_{j+1}^2(0)\,\{h_{j+1}^2 - 2\,h_j\,h_{j+1}\,\cos(\phi_j) + h_j^2\} + \{\Delta h_j\,u'_{j+1}(0) + \Delta u'_j\,h_j\}^2}\,. \qquad [17]$$

For the quantities u, u' and h, the subscript j denotes the conditions at the beginning of the j[th] subarc, and on the variables $\Delta u', \Delta h$ and ϕ the subscript j denotes the j[th] impulse which occurs at the end of the j[th] subarc. In addition the following equality and inequality constraints must be satisfied:

Equality Constraints

The conditions for rendezvous require the following position and velocity matching constraints:

i. final position match constraint -

$$h_1(x) \equiv \vec{r}_f - \vec{r}_t(\eta_t) = 0, \qquad [18]$$

ii. final velocity match constraint -

$$h_2(x) \equiv \quad \vec{v}_f \; - \; \vec{v}_t(\eta_t) = 0, \qquad [19]$$

iii. time of flight constraint -

$$h_3(x) \equiv \quad T_f \; - \; T_t \; = \; 0, \qquad [20]$$

where the subscript f refers to the conditions on the interceptor trajectory after the final impulse and the subscript t refers to conditions on the target.

Inequality Constraints

Additional constraints which must be avoided along each arc of the interceptor or target trajectory in the form of an inequality are:

i. non-negativity of the coasting arcs of the interceptor -

$$g_i(x) \equiv \quad \eta_i \geq 0 \qquad \mathrm{i} \; = \; 1, \ldots, \mathrm{nim}, \qquad [21]$$

ii. non-negativity of the coasting arc of the target -

$$g_{\mathrm{nim}+1}(x) \equiv \quad \eta_t \geq 0, \qquad [22]$$

iii. time of flight limit constraint (maximum time specified for rendezvous) -

$$g_{\mathrm{nim}+2}(x) \equiv \quad T_{\max} \; - \; T_f \geq 0, \qquad [23]$$

iv. minimum radius constraint for each coasting arc except the initial coast arc of the interceptor trajectory -

$$g_j(x) \equiv \quad u_0 - u_{\max} \geq 0 \quad j \; = \; \mathrm{nim} + 3, \ldots, 2\,\mathrm{nim} + 1. \qquad [24]$$

The transfer arc should lie outside a circle of radius $r_0 \equiv \frac{1}{u_0}$. This is essentially a semi-infinite constraint. But from the nature of the transfer arcs, i.e., conic sections, the minimum radius on any subarc is given by:

$$\frac{1}{u_{\max}} = \begin{cases} \text{perigee radius, if perigee passage occurs on subarc,} \\ \min \ (r_{\text{initial}}, r_{\text{final}}), \text{ otherwise} \end{cases}$$

The minimum radius constraint as given above is not C^2, consequently for the moment we have chosen a stiffer constraint of requiring the perigee radius of any transfer arc to be greater than than the minimum allowable radius.

v. non-negativity of the radius constraint -

$$g_j(x) \equiv \quad u_{\min} \geq 0 \quad j = 2\,\text{nim} + 2, \dots, 3\,\text{nim}. \qquad [25]$$

This too is a semi-infinite constraint, and here we require the final radius to be positive. This constraint is considered to disallow the possibility of negative distances which are mathematically possible from the nature of its governing equations.

The known problem, the solution to which was obtained from earlier work is given by:

i. specified number of impulses for the interceptor = 4;

ii. the interceptor resides initially on a circular orbit of radius = 1.20 DU;

iii. the target resides on a circular orbit of radius = 1.45 DU;

iv. the phase difference angle between the target and the interceptor at the start of the maneuver ≤ 180. deg;

v. time of flight limit for the rendezvous = 160. min;

vi. minimum allowable radius (1 DU = radius of earth) = 0.90 DU.

In all for this 4 impulse problem there are 17 variables and the total number of constraints are: 7 equality and 12 inequality constraints. For the above mentioned conditions, the time of flight limit inequality constraint [23] and the constraint [21] on the initial coasting arc are active. The trajectory for this problem is shown in Figure 3.

Starting with the solution to this problem, we solve a problem for which all conditions are the same except that now we require the minimum allowable radius to be $r_0 = 1.0$ DU and the time of flight to be $T_{max} = 150$ min. For the solution obtained, the above mentioned active constraints still remain active, and the minimum radius constraint on the third subarc becomes active. For an accuracy of 10^{-12}, the curve tracking procedure required 77 Jacobian evaluations, and the arc length of the connecting path was 1.528377. The trajectory so obtained is shown in Figure 4. The solution obtained is in accordance with the results obtained in a prior work[14].

Remarks

We have applied the Chow-Yorke algorithm to solve constrained parameter optimization problems. In this regard, one has to choose both an approriate homotopy map and an appropriate curve tracking procedure. The success of this algorithm lies mainly in finding the appropriate map connecting the initial (solved) problem and the given (unsolved) problem. Finding such a map is unfortunately problem class dependent and so a universal procedure is not advocated, although maps can be found for classes of problems. We have provided an alternative to the active constraint set strategy, wherein it is no longer necessary to monitor constraints and their multipliers to switch constraints. Here we assume that at the end of the homotopy path the complementarity conditions hold strictly and that if a constraint is active, it does not have a zero gradient. At the end of the homotopy path, the appropriate constraints become active automatically, and we have a candidate solution which satisfies the Kuhn-Tucker necessary conditions.

Acknowledgements

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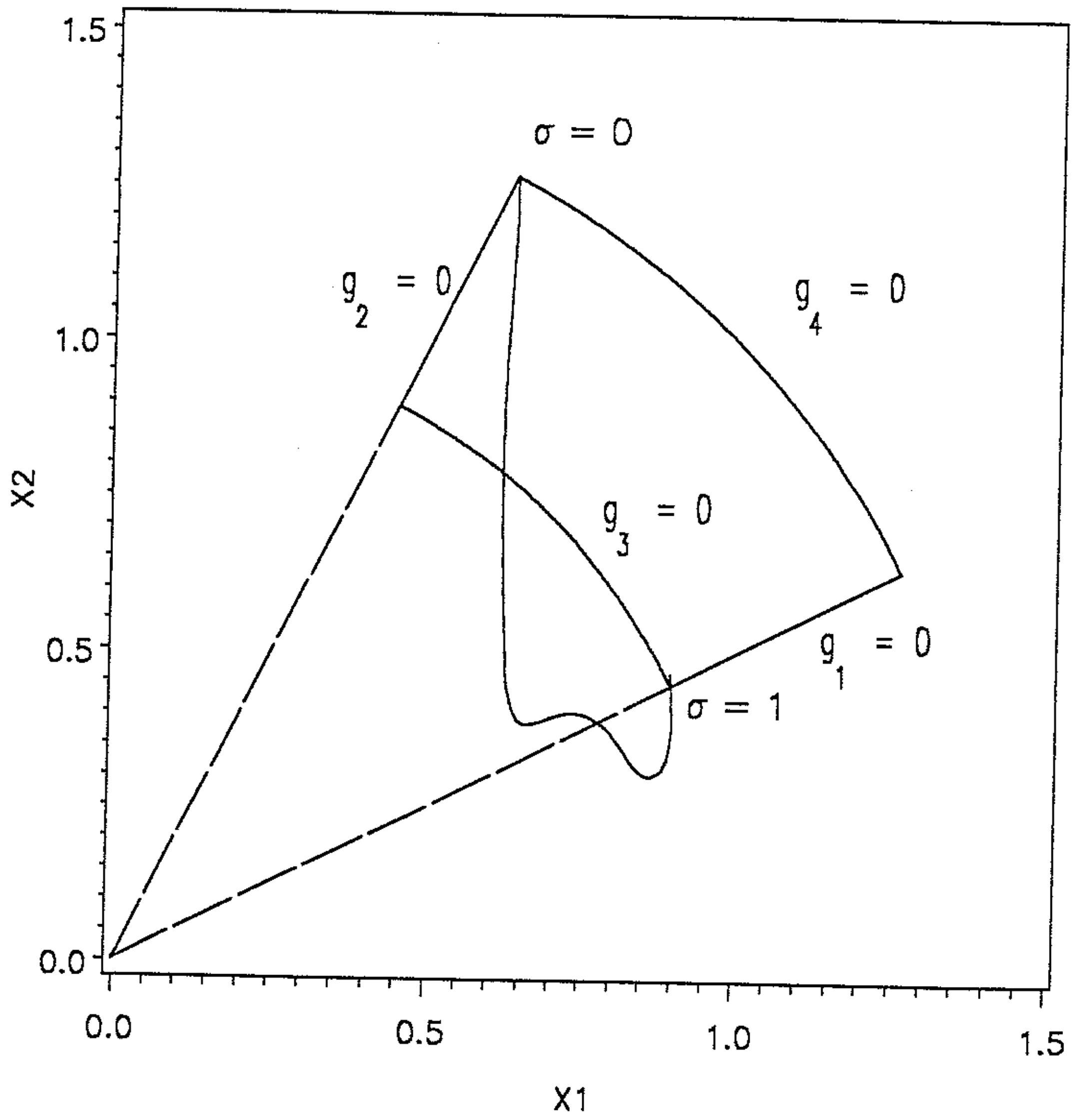

Figure 1. Feasible region for Example 1 and the plot of x_1 versus x_2 with respect to the variation in σ.

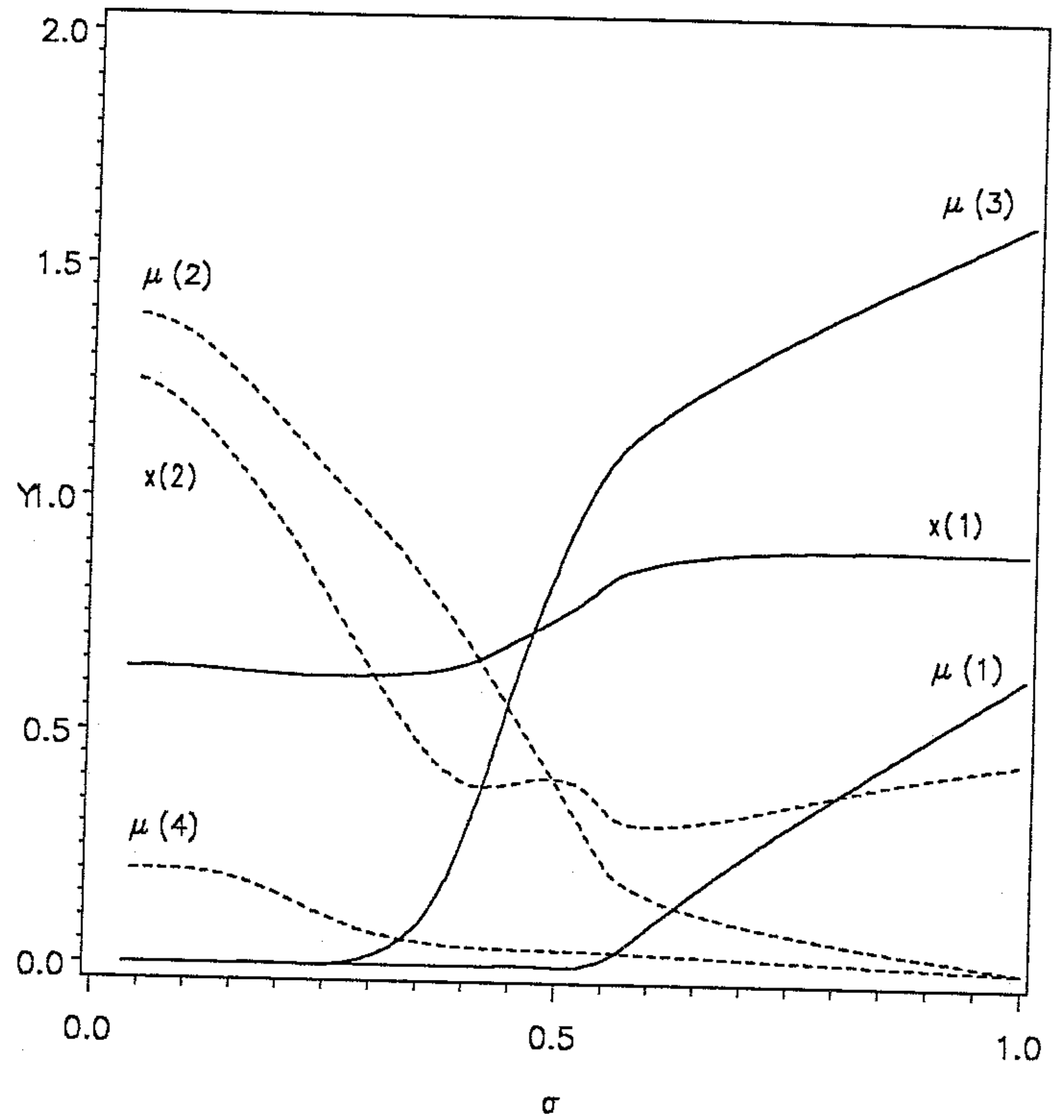

Figure 2. Variation of Y = $\{x_1, x_2, \mu_1, \mu_2, \mu_3, \mu_4\}$ with respect to σ.

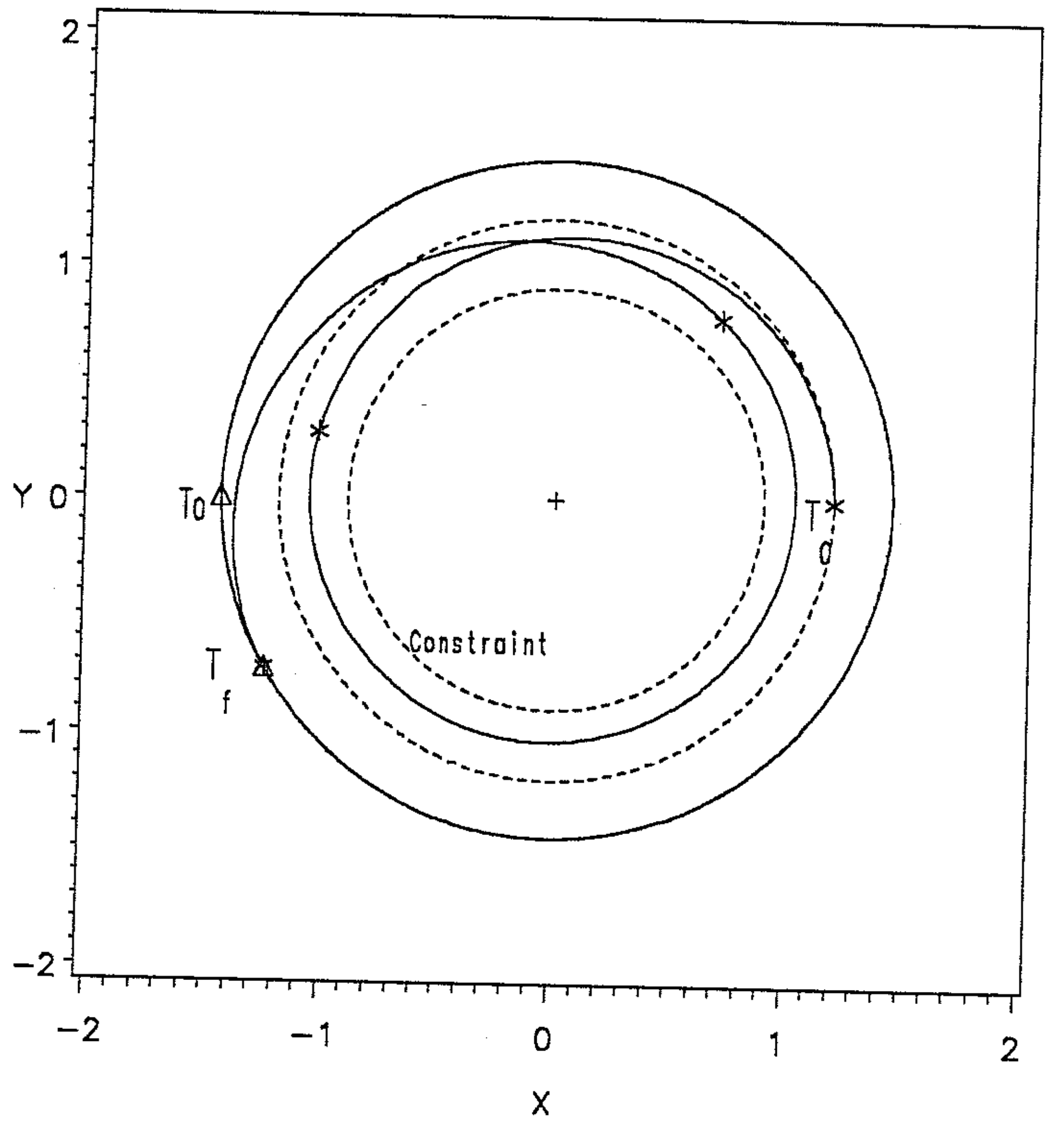

Figure 3. 4 impulse trajectory, with tof = 160 mins, and minimum allowable radius $r_0 = 0.9\mathrm{DU}$.

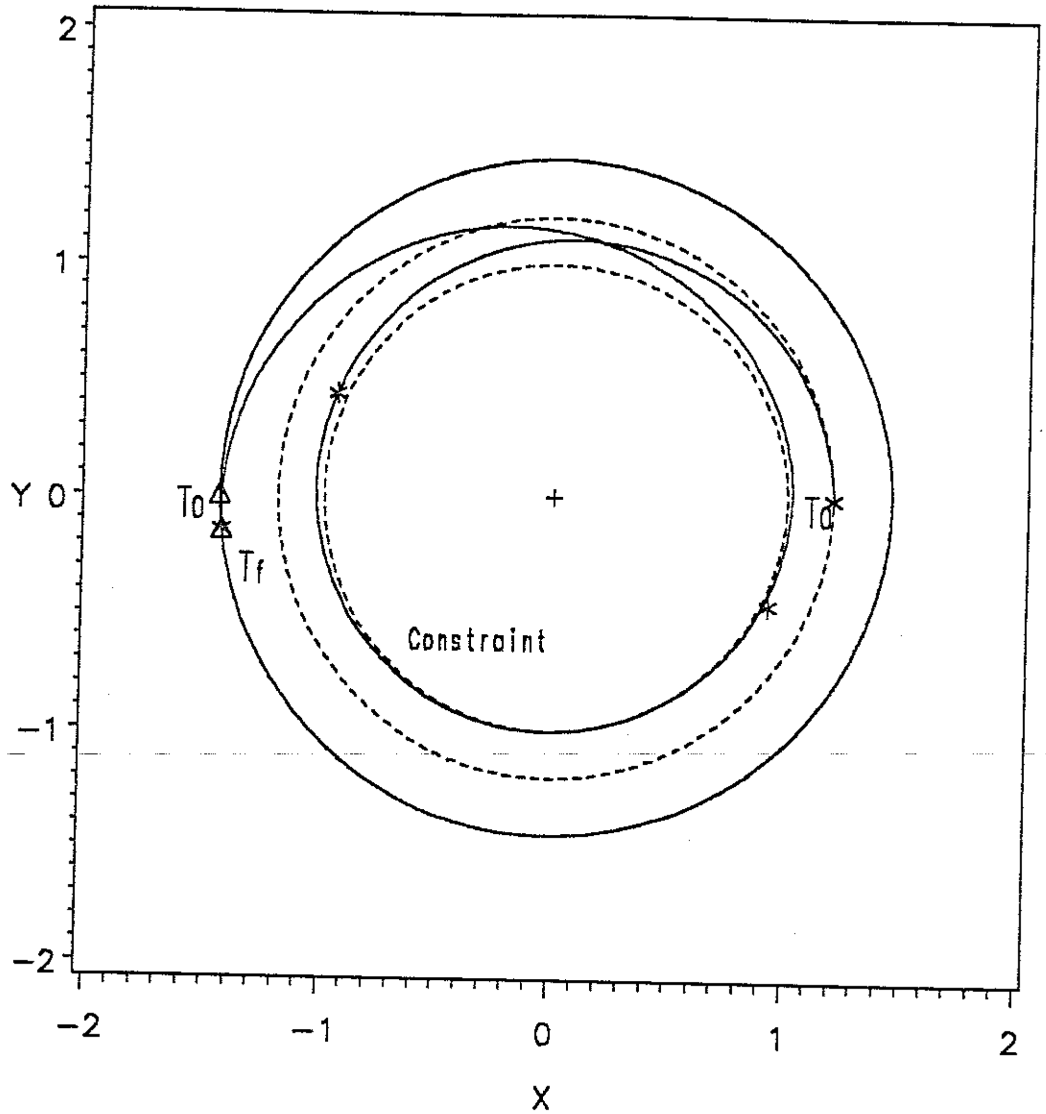

Figure 4. 4 impulse trajectory, with tof = 150 mins, and minimum allowable radius $r_0 = 1.0$DU.