Least Change Secant Update Methods for Undetermined Systems

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LEAST CHANGE SECANT UPDATE METHODS FOR UNDERDETERMINED SYSTEMS*

HOMER F. WALKER† AND LAYNE T. WATSON‡

Abstract. Least-change secant updates for nonsquare matrices have been addressed recently in [6]. Here we consider the use of these updates in iterative procedures for the numerical solution of underdetermined systems. Our model method is the normal flow algorithm used in homotopy or continuation methods for determining points on an implicitly defined curve. A Kantorovich-type local convergence analysis is given which supports the use of least-change secant updates in this algorithm. This analysis also provides a Kantorovich-type local convergence analysis for least-change secant update methods in the usual case of an equal number of equations and unknowns. This in turn gives a local convergence analysis for augmented Jacobian algorithms which use least-change secant updates. We conclude with the results of some numerical experiments.

Key words. underdetermined systems, least-change secant update methods, quasi-Newton methods, normal flow algorithm, augmented Jacobian matrix algorithm, continuation methods, homotopy methods, curve-tracking algorithms, parameter-dependent systems

AMS(MOS) subject classifications. 65H10

1. Introduction. Our notational conventions, which are not strictly observed but are intended to serve as helpful guidelines for remembering what is what, are the following: Unless otherwise indicated, lower case letters denote vectors and scalars, and capital letters denote matrices and operators. Boldface upper case letters denote vector spaces, subspaces, and affine subspaces. For positive integers p and q, $\mathbf{R}^p$ denotes p-dimensional real Euclidean space and $\mathbf{R}^{p\times q}$ denotes the space of real $p \times q$ matrices. We refer particularly to $\mathbf{R}^n$ and $\mathbf{R}^{\bar{n}}$ for $\bar{n} \geq n$, and for convenience, we set $\bar{n} = n + m$ for $m \geq 0$. Vectors with bars are in $\mathbf{R}^{\bar{n}}$; without bars, they are in $\mathbf{R}^n$ or $\mathbf{R}^m$ unless otherwise indicated. We often partition vectors, e. g., we write $\bar{x} \in \mathbf{R}^{\bar{n}}$ as $\bar{x} = (x, \lambda)$ for $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}^m$, and we do not distinguish between (x,λ) and $\begin{pmatrix} x \\ \lambda \end{pmatrix}$. We also often partition matrices, e. g., we write $B \in \mathbf{R}^{n\times\bar{n}}$ as $B = [\mathcal{B}, \mathcal{C}]$ for $\mathcal{B} \in \mathbf{R}^{n\times n}$ and $\mathcal{C} \in \mathbf{R}^{n\times m}$. The dimensions of vector and matrix partitions are made clear in each case, usually by the context. We use "Jacobian" to mean "Jacobian matrix", and we denote the full Jacobian of a function F by F'. If F is

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a function of $\bar{x} = (x, \lambda) \in \mathbf{R}^{\bar{n}}$, then we denote partial Jacobians $\partial F/\partial x$ by F_x, $\partial F/\partial \lambda$ by F_λ, etc. We assume throughout the following that there are given but unspecified vector norms on $\mathbf{R}^n$, $\mathbf{R}^m$, and $\mathbf{R}^{\bar{n}}$, together with their associated induced matrix norms, and we denote all of these norms by $|\cdot|$. Similarly, we assume there is a given but unspecified matrix norm on $\mathbf{R}^{n\times\bar{n}}$ associated with a matrix inner product, and we denote this norm by $\|\cdot\|$. A projection onto a subspace or affine subspace which is orthogonal with respect to $\|\cdot\|$ is denoted by P with the subspace or affine subspace appearing as a subscript. If P denotes a projection, then we set $P^\perp = I - P$, where I is the identity operator.

Of interest here is the numerical solution of a zero-finding problem for a (possibly) underdetermined nonlinear system, which we write in the following form.

PROBLEM 1.1. *Given $F : \mathbf{R}^{\bar{n}} \to \mathbf{R}^n$ with $\bar{n} \geq n$, find $\bar{x}_* \in \mathbf{R}^{\bar{n}}$ such that $F(\bar{x}_*) = 0$.*
We make the following basic hypothesis throughout the sequel.

HYPOTHESIS 1.2. *F is differentiable and F' is of full rank n in an open convex set Ω, and the following hold:*

(i) there exist $\gamma \geq 0$ and $p \in (0, 1]$ such that $|F'(\bar{y}) - F'(\bar{x})| \leq \gamma\, |\bar{y} - \bar{x}|^p$ for all $\bar{x}$, $\bar{y} \in \Omega$;

(ii) there is a constant μ for which $|F'(\bar{x})^+| \leq \mu$ for all $\bar{x} \in \Omega$.

In Hypothesis 1.2, the superscript "+" indicates pseudo-inverse. That is, for $b \in \mathbf{R}^n$ and $\bar{x} \in \Omega$, $F'(\bar{x})^+ b \in \mathbf{R}^{\bar{n}}$ is the solution of $F'(\bar{x})\bar{s} = b$ having minimal norm, i. e., the solution which is orthogonal to the null-space of $F'(\bar{x})$, i. e., the solution which is in the span of the columns of $F'(\bar{x})^T$. For the analysis in the following, we also define for $\eta > 0$

$$\Omega_\eta = \{\bar{x} \in \Omega : |\bar{y} - \bar{x}| < \eta \Rightarrow \bar{y} \in \Omega\}. \tag{1.1}$$

Problems such as Problem 1.1 arise in a variety of contexts. One is equality-constrained optimization, in which Problem 1.1 is the problem of finding a point on a constraint surface. Another is parameter-dependent systems of nonlinear equations, in which usually $\bar{x} = (x, \lambda)$, where $x \in \mathbf{R}^n$ is an independent variable and $\lambda \in \mathbf{R}^m$ is a parameter vector. Of particular interest here is the context of homotopy or continuation methods for determining points on an implicitly defined curve, in which one has $\bar{n} = n + 1$ and $\bar{x} = (x, \lambda)$ with $\lambda \in \mathbf{R}^1$. For a description of these methods, see the extensive survey of Allgower and Georg [2] and also Georg [15], Morgan [19], [20], Rheinboldt [23], and Watson and Fenner [24], Watson [25]-[28], Watson and Scott [29], Watson and Scott [30], and Watson, Billups, and Morgan [31]. Here, we consider arbitrary $\bar{n} \geq n$ since doing so incurs no additional difficulty, offers important advantages in the sequel, and is useful for the full range of applications.

Problem 1.1 must generally be solved numerically by some iterative method. The model method here is the normal flow algorithm [16] used in homotopy or continuation methods; see, e. g., Watson, Billups, and Morgan [31] and the references given there. We write this model method as

ALGORITHM 1.3. *Given $\bar{x}_0 \in \mathbf{R}^{\bar{n}}$, determine for $k = 0, 1, \ldots,$*

$$\bar{x}_{k+1} = \bar{x}_k - F'(\bar{x}_k)^+ F(\bar{x}_k).$$

Algorithm 1.3 takes the name "normal flow" from the $\bar{n} = n+1$ case, in which the iteration steps are asymptotically normal to the *Davidenko flow*; see [7] and [31]. For any $\bar{n}$, it is clear that each iteration step $-F'(\bar{x}_k)^+F(\bar{x}_k)$ is normal to the manifold $F(\bar{x}) \equiv F(\bar{x}_k)$. Of course Algorithm 1.3 is just Newton's method in the $\bar{n} = n$ case. As with Newton's method in the $\bar{n} = n$ case, it may be necessary in practice to augment Algorithm 1.3 and all other algorithms considered below with procedures for modifying the iteration step to ensure progress from bad starting points, but we need not consider such procedures here. Algorithm 1.3 also shares with Newton's method the computational expense of evaluating the Jacobian and solving a linear system for the step at each iteration, and this expense is especially likely to be significant when the dimension of the system is large.

In the $\bar{n} = n$ case, *quasi-Newton methods* are very widely used as cost-effective alternatives to Newton's method. The basic form of a quasi-Newton method for solving $F(x) = 0$, $F : \mathbf{R}^n \to \mathbf{R}^n$, is

(1.2) $$x_{k+1} = x_k - B_k^{-1}F(x_k),$$

in which $B_k \approx F'(x_k) \in \mathbf{R}^{n\times n}$, the Jacobian of F at x_k. The most generally effective quasi-Newton methods are those in which each successive B_{k+1} is determined as a *least-change secant update* of its predecessor B_k. As the name suggests, one determines B_{k+1} as a least-change secant update of B_k by making the least possible change in B_k (as measured by a suitable matrix norm) which incorporates current secant information (usually expressed in terms of successive x- and F-values) and other available information about the structure of F'. There are also notable updates which, strictly speaking, are least-change *inverse* secant updates obtained in an analogous way by making the least possible change to B_k^{-1}. When speaking generically of least-change secant updates, we intend to include these. When distinguishing least-change secant updates from least-change inverse secant updates, we sometimes refer to the former as *direct* least-change secant updates. In [12], Dennis and Schnabel precisely formalize the notions associated with least-change secant updates and show how the updates most widely used in quasi-Newton methods can be derived as least-change secant updates. In [14], Dennis and Walker show that *least-change secant update methods*, i. e., quasi-Newton methods using least-change secant updates, can be expected to have desirable convergence properties in general. See also Dennis and Schnabel [13] as a general reference on all aspects of quasi-Newton and least-change secant update methods.

In view of the success of least-change secant update methods in the $\bar{n} = n$ case, it is natural to consider least-change secant update methods for general $\bar{n} \geq n$ which are obtained from Algorithm 1.3 by replacing $F'(\bar{x}_k)$ with a matrix maintained by least-change secant updating. The main purpose of this paper is to study such algorithms.

In §2 below, we consider Algorithm 1.3 and analogous algorithms which use least-change secant updates. For the record and to set the stage for further analysis, we first give a local convergence theorem for Algorithm 1.3. Our understanding is that something like this local convergence result has been assumed in folklore but has not been previously published [1], although some results for a modified version of Algorithm 1.3 have been given by Ben-Israel [4]. Next, we formulate and develop a local convergence analysis for analogues of Algorithm 1.3 which use nonsquare-matrix extensions of least-change secant and inverse-secant updates given recently by Bourji and Walker [6] and Beattie and Weaver-Smith [3]. We note that these and all other updating algorithms considered in this paper are, in the terminology of [14], *fixed-scale* least-change secant update methods. That is, the norm $\|\cdot\|$ on $\mathbf{R}^{n\times\bar{n}}$ used to define least-change secant updates remains the same for all iterations. Thus our analysis does not apply to algorithms which use the nonsquare-matrix extensions of the Davidon-Fletcher-Powell and Broyden-Fletcher-Goldfarb-Shanno updates given in [6], for these updates are least-change with respect to norms which vary from one iteration to the next.

In [6], a local convergence analysis is given for certain paradigm iterations for solving Problem 1.1 which use least-change secant updates. Although these paradigm iterations are very general in some ways and more or less include the updating algorithms given in this paper, the local convergence analysis in [6] does not apply to the algorithms here. Indeed, the local convergence results of [6] are conditioned on the rate of convergence of the last m components of the iterates to their limits, and nothing can be said about this rate of convergence for the updating algorithms given here.

The analysis in §2 proceeds more or less along standard lines in many ways, and the developments parallel those of [14] and [6] in many particulars. We have followed the usual approach (cf. [14], [6]) of carrying out most of the difficult technical work in a very general context and isolating the details in an appendix. However, the analysis of §2 does have the important, somewhat nontraditional feature of being a *Kantorovich-type analysis*; see, e. g., [22]. By this we mean that there is no a priori assumption of existence of or proximity to a solution of Problem 1.1 which is expected to be a limit of an iteration sequence. One is forced to resort to such an analysis in the context of interest here, since solutions of Problem 1.1 cannot be assumed to be isolated and therefore no particular solution can be singled out a priori as an expected limit of an iteration sequence. We hasten to note that our analysis does not use the method of "majorization", which some regard as characteristic of a Kantorovich-type analysis (cf. Marwil [17]), but accomplishes the same ends through more direct means. We also note that with $\bar{n} = n$, this analysis provides a Kantorovich-type local convergence analysis for general fixed-scale least-change secant and inverse secant update methods in the usual case of an equal number of equations and unknowns. Kantorovich-type local convergence analyses (using "majorization") have previously been given in the $\bar{n} = n$ case for least-change secant update methods which use Broyden or sparse Broyden updates by Dennis [9], Marwil [17], and Dennis and Li [11] and for more general quasi-Newton methods of the form (1.2) by Dennis [8], [10].

An iterative method other than Algorithm 1.3 which is often used in homotopy or continuation methods is the *augmented Jacobian algorithm*; see, e. g., Billups [5], Georg [15], Rheinboldt [23], and Watson, Billups, and Morgan [31]. We consider this method in the following basic form:

ALGORITHM 1.4. *Given $\bar{x}_0 \in \mathbf{R}^{\bar{n}}$ and $V \in \mathbf{R}^{m \times \bar{n}}$ such that $\begin{bmatrix} F'(\bar{x}_0) \\ V \end{bmatrix}$ is nonsingular, determine for $k = 0, 1, \ldots,$*

$$\bar{x}_{k+1} = \bar{x}_k + \bar{s}_k,$$

where $\bar{s}_k$ satisfies $F'(\bar{x}_k)\bar{s}_k = -F(\bar{x}_k)$ and $V\bar{s}_k = 0$.

Other forms of this algorithm are considered in the $\bar{n} = n + 1$ case in [15], [23], and [31], including forms in [15] and [31] which use a simple least-change secant update (the (first) Broyden update, see [6] and §4 below) to approximate F'. In [15] and [23], V is taken to be the transpose of a well-chosen unit basis vector in $\mathbf{R}^{\bar{n}}$; in [31], V is taken to be the transpose of an approximate tangent vector to the solution curve.

In §3 below, we first use the results of §2 to outline a local convergence analysis for Algorithm 1.4 and for an analogue which uses direct least-change secant updates to approximate F'. The approach is to embed the system of Problem 1.1 in an augmented system of $\bar{n}$ equations in a natural way and then to apply the results of §2 in the case of an equal number of equations and unknowns. We then formulate local convergence results for an analogue of Algorithm 1.4 which uses least-change inverse secant updates, sketching proofs which parallel those of the corresponding results in §2.

In §4, we outline some numerical experiments. These experiments are not intended to be at all exhaustive or conclusive but rather to indicate some basic properties of and issues associated with the methods considered here.

2. The normal flow algorithm.

We begin with a local convergence theorem for Algorithm 1.3.

THEOREM 2.1. *Let F satisfy Hypothesis 1.2 and suppose Ω_η is given by (1.1) for some $\eta > 0$. Then there is an $\epsilon > 0$ depending only on γ, p, μ, and η such that if $\bar{x}_0 \in \Omega_\eta$ and $|F(\bar{x}_0)| < \epsilon$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm 1.3 are well-defined and converge to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$. Furthermore, there is a constant β for which*

$$|\bar{x}_{k+1} - \bar{x}_*| \leq \beta |\bar{x}_k - \bar{x}_*|^{1+p}, \qquad k = 0, 1, \ldots. \tag{2.1}$$

Proof. If $\bar{x} \in \Omega$ and $\bar{s} = -F'(\bar{x})^+ F(\bar{x})$, then

$$|\bar{s}| \leq \mu |F(\bar{x})|. \tag{2.2}$$

If also $\bar{x}_+ = \bar{x} + \bar{s} \in \Omega$, then Proposition A.3 in the Appendix (with $\bar{y} = \bar{x}_+$ and $B = F'(\bar{x})$) gives

$$|F(\bar{x}_+)| = |F(\bar{x}_+) - F(\bar{x}) - F'(\bar{x})\bar{s}| \leq \frac{\gamma}{1+p} |\bar{s}|^{1+p}. \tag{2.3}$$

If $\bar{s}_+ = -F'(\bar{x}_+)^+F(\bar{x}_+)$, then (2.2) and (2.3) give

$$|\bar{s}_+| \leq \frac{\gamma\mu}{1+p}|\bar{s}|^{1+p}. \tag{2.4}$$

Suppose $\epsilon > 0$ is so small that

$$\frac{\gamma\mu^{1+p}\epsilon^p}{1+p} < 1 \qquad \text{and} \qquad \frac{\mu\epsilon}{1-\left(\frac{\gamma\mu^{1+p}\epsilon^p}{1+p}\right)} < \eta.$$

It follows from (2.2) and (2.4) by an easy induction that if $\bar{x}_0 \in \Omega_\eta$ and $|F(\bar{x}_0)| < \epsilon$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm 1.3 are well-defined, remain in Ω, and constitute a Cauchy sequence with limit $\bar{x}_* \in \Omega$. It follows from (2.3) that $F(\bar{x}_*) = 0$.

To complete the proof, we show that if $\{\bar{x}_k\}_{k=0,1,\ldots}$ is determined by Algorithm 1.3 and converges to $\bar{x}_* \in \Omega$, then (2.1) holds for an appropriate β. From (2.2) and (2.3), one has

$$|F(\bar{x}_{k+1})| \leq \frac{\gamma\mu^{1+p}}{1+p}|F(\bar{x}_k)|^{1+p}, \qquad k = 0, 1, \ldots. \tag{2.5}$$

Proposition A.3 (with $\bar{x} = \bar{x}_*$, $\bar{y} = \bar{x}_k$, and $B = F'(\bar{x}_*)$) gives

$$|F(\bar{x}_k)| \leq |F'(\bar{x}_*)||\bar{x}_k - \bar{x}_*| + \frac{\gamma}{1+p}|\bar{x}_k - \bar{x}_*|^{1+p}$$

for each k, and so (2.5) yields

$$|F(\bar{x}_{k+1})| \leq \beta'|\bar{x}_k - \bar{x}_*|^{1+p}, \qquad k = 0, 1, \ldots, \tag{2.6}$$

for an appropriate β'. Setting $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k = -F'(\bar{x}_k)^+F(\bar{x}_k)$ for each k, one has from (2.2) that

$$|F(\bar{x}_{k+1})| \geq \frac{|\bar{s}_{k+1}|}{\mu}, \qquad k = 0, 1, \ldots. \tag{2.7}$$

Also, (2.4) gives

$$\begin{aligned}
|\bar{s}_{k+1}| &\geq |\bar{x}_{k+1} - \bar{x}_*| - |\bar{x}_{k+2} - \bar{x}_*| \\
&\geq |\bar{x}_{k+1} - \bar{x}_*| - \sum_{j=k+2}^{\infty} |\bar{s}_j| \\
&\geq |\bar{x}_{k+1} - \bar{x}_*| - \frac{\gamma\mu}{1+p}\sum_{j=k+1}^{\infty} |\bar{s}_j|^{1+p} \\
&\geq |\bar{x}_{k+1} - \bar{x}_*| - \left\{\frac{\gamma\mu}{1+p}\sum_{j=k+1}^{\infty} |\bar{s}_j|^p\right\}|\bar{s}_{k+1}|
\end{aligned}$$

for k so large that $|\bar{s}_{k+1}| \geq |\bar{s}_j|$ for $j \geq k+1$, with all sums finite. Then

$$|\bar{s}_{k+1}| \geq \beta''|\bar{x}_{k+1} - \bar{x}_*|, \qquad k = 0, 1, \ldots, \tag{2.8}$$

for an appropriate $\beta'' > 0$. It follows from (2.6), (2.7), and (2.8) that (2.1) holds for an appropriate β. □

We now introduce analogues of Algorithm 1.3 which use the nonsquare-matrix least-change secant updates in [6] and [3]. For completeness, we very briefly review the general definitions of these updates here before introducing the algorithms which use them. For more discussion and, in particular, for specific formulas which extend the well-known square-matrix updates to the nonsquare case, see [6] and [3].

Throughout the following, we assume an affine subspace $\mathbf{A} \subseteq \mathbf{R}^{n\times\bar{n}}$ is given in which updated matrices are to lie. The elements of $\mathbf{A}$ are presumed to reflect structure of F' which one wants to impose on Jacobian approximations. We denote the parallel subspace of $\mathbf{A}$ by $\mathbf{S}$. For our definitions, we suppose that there are also given a matrix $B \in \mathbf{R}^{n\times\bar{n}}$ to be updated and *secant information* in the form of vectors $\bar{s} \in \mathbf{R}^{\bar{n}}$ and $y \in \mathbf{R}^n$. We set $\mathbf{Q}(y, \bar{s}) = \{M \in \mathbf{R}^{n\times\bar{n}} : M\bar{s} = y\}$ and note that $\mathbf{Q}(y,\bar{s})$ is an affine subspace with parallel subspace $\mathbf{N}(\bar{s}) = \{M \in \mathbf{R}^{n\times\bar{n}} : M\bar{s} = 0\}$. We define $\mathbf{M}(\mathbf{A}, \mathbf{Q}(y,\bar{s}))$ to be the set of elements of $\mathbf{A}$ which are closest to $\mathbf{Q}(y,\bar{s})$ in the norm $\|\cdot\|$ if $\mathbf{Q}(y,\bar{s}) \neq \emptyset$; otherwise, we set $\mathbf{M}(\mathbf{A}, \mathbf{Q}(y,\bar{s})) = \mathbf{A}$. Of course, $\mathbf{M}(\mathbf{A}, \mathbf{Q}(y,\bar{s})) = \mathbf{A} \cap \mathbf{Q}(y,\bar{s})$ if $\mathbf{A} \cap \mathbf{Q}(y,\bar{s}) \neq \emptyset$.

We make the following

DEFINITION 2.2. *$B_+ \in \mathbf{R}^{n\times\bar{n}}$ is the least-change secant update of B in $\mathbf{A}$ with respect to $\bar{s}$, y, and the norm $\|\cdot\|$ if B_+ is the unique solution of*

$$\min_{\tilde{B}\in\mathbf{M}(\mathbf{A},\mathbf{Q}(y,\bar{s}))} \left\|\tilde{B} - B\right\|.$$

Particular least-change secant updates to which the analysis below is relevant are the following, all obtained with $\|\cdot\| = \|\cdot\|_F$, the Frobenius norm:

(i) the (first) Broyden update of [6], obtained with $\mathbf{A} = \mathbf{R}^{n\times\bar{n}}$;

(ii) the Powell symmetric Broyden update of [6], obtained with $\mathbf{A}$ equal to the set of matrices such that some particular subset of n columns exhibits symmetry;

(iii) the sparse Broyden update of [6], obtained with $\mathbf{A}$ equal to the set of matrices having a particular pattern of sparsity;

(iv) the sparse symmetric update of [3], obtained with $\mathbf{A}$ equal to the set of matrices having a particular pattern of sparsity and such that some particular subset of n columns exhibits symmetry.

To define a least-change inverse secant update, we assume that B is of full rank n. As in [6], we assume in particular that the first n columns of B constitute a nonsingular matrix, although we stress that any set of n linearly independent columns of B can be used instead. We write $B = [\mathcal{B}, \mathcal{C}]$ for nonsingular $\mathcal{B} \in \mathbf{R}^{n\times n}$ and for $\mathcal{C} \in \mathbf{R}^{n\times m}$ and set $K = [\mathcal{K}, \mathcal{L}]$, where $\mathcal{K} = \mathcal{B}^{-1}$ and $\mathcal{L} = -\mathcal{B}^{-1}\mathcal{C}$. We then make the following

DEFINITION 2.3. *$K_+ \in \mathbf{R}^{n\times\bar{n}}$ is the least-change inverse secant update of K in $\mathbf{A}$ with respect to $\bar{s}$, y, and the norm $\|\cdot\|$ if K_+ is the unique solution of*

$$\min_{\tilde{K}\in\mathbf{M}(\mathbf{A},\mathbf{Q}(s,\bar{y}))} \left\|\tilde{K}-K\right\|,$$

where $\bar{s}=(s,t)$ for $s\in\mathbf{R}^n$ and $t\in\mathbf{R}^m$ and $\bar{y}=(y,t)$.

From $K_+ = [\mathcal{K}_+,\mathcal{L}_+]$, one can obtain $B_+ = [\mathcal{B}_+,\mathcal{C}_+]$ by taking $\mathcal{B}_+ = \mathcal{K}_+^{-1}$ and $\mathcal{C}_+ = -{\mathcal{K}_+}^{-1}\mathcal{L}_+$; we also refer to this B_+ as the least-change inverse secant update of B. Particular least-change inverse secant updates to which the analysis below is relevant are the following, all obtained with $\|\cdot\| = \|\cdot\|_F$:

(i) the second Broyden update of [6], obtained with $\mathbf{A}=\mathbf{R}^{n\times\bar{n}}$;

(ii) the Greenstadt update of [6], obtained with $\mathbf{A}$ equal to the set of matrices such that some particular subset of n columns is nonsingular and exhibits symmetry.

We note that whether B_+ is obtained as a direct or inverse least-change secant update of B, B_+ satisfies the *secant equation* $B_+\bar{s}=y$ if at all possible, i. e., if $\mathbf{A}\cap\mathbf{Q}(y,\bar{s})\neq\emptyset$ in the direct update case or if $\mathbf{A}\cap\mathbf{Q}(s,\bar{y})\neq\emptyset$ in the inverse update case. In any event, B_+ is as close as possible to a matrix satisfying the secant equation among the set of allowable Jacobian approximants. We also note for the analysis in the sequel that if B_+ is a direct least-change secant update of B in $\mathbf{A}$, then for any $G\in\mathbf{M}(\mathbf{A},\mathbf{Q}(y,\bar{s}))$,

$$B_+ = P^{\perp}_{\mathsf{S}\cap\mathbf{N}(\bar{s})}G + P_{\mathsf{S}\cap\mathbf{N}(\bar{s})}B \tag{2.9}$$

and for any $M\in\mathbf{R}^{n\times\bar{n}}$

$$\|B_+ - M\| \le \left\|P^{\perp}_{\mathsf{S}\cap\mathbf{N}(\bar{s})}(G-M)\right\| + \left\|P_{\mathsf{S}\cap\mathbf{N}(\bar{s})}(B-M)\right\|; \tag{2.10}$$

see [14] and [6]. Analogous expressions hold for least-change inverse secant updates.

In addition to the updates of Definitions 2.2 and 2.3, one might think it natural to consider a *least-change pseudo-inverse secant update* obtained by making a minimal norm change in the pseudo-inverse to obtain a matrix with a pseudo-inverse as close as possible to that of a matrix satisfying the secant equation. It is straightforward to define such an update, and it has the appeal of offering an extension of the usual square-matrix least-change inverse secant update which, in contrast to the update of Definition 2.3, does not depend on a particular set of linearly independent columns of the matrix being updated. However, we do not think such an update is likely to be successful. At least it is clear that the local convergence analysis given below cannot be extended to apply to an algorithm which uses such an update. Crucial to such an extension would be a *bounded deterioration* property analogous to Hypotheses A.2 and A.8, which would ensure local linear convergence, and a pseudo-inverse update counterpart of the inequality (2.20) below, which would allow one to show superlinear or optimal linear convergence. Consideration of the pseudo-inverse analogue of the Broyden update suggests that these properties cannot be expected to hold for least-change pseudo-inverse secant updates. Indeed, for $B \in \mathbf{R}^{n \times \bar{n}}$ and $K = B^+ \in \mathbf{R}^{\bar{n} \times n}$, the update of K which is the pseudo-inverse analogue of the Broyden update is obtained by taking $\|\cdot\| = \|\cdot\|_F$ and $\mathbf{A} = \mathbf{R}^{\bar{n} \times n}$, which gives

$$K_+ = P_{\mathbf{N}(y)} K + \frac{\bar{s} y^T}{y^T y} = K + \frac{(\bar{s} - Ky) y^T}{y^T y},$$

where $\mathbf{N}(y) = \{M \in \mathbf{R}^{\bar{n} \times n} : My = 0\}$. In the usual case, $\bar{s} = \bar{x}_+ - \bar{x}$ and $y = F(\bar{x}_+) - F(\bar{x})$ for some $\bar{x}$, $\bar{x}_+$, and one has

$$\begin{aligned} K_+ - F'(\bar{x}_+)^+ &= K_+ - F'(\bar{x})^+ - \left[F'(\bar{x}_+)^+ - F'(\bar{x})^+\right] \\ &= P_{\mathbf{N}(y)} \left[K - F'(\bar{x})^+\right] + \frac{\left[I - F'(\bar{x})^+ F'(\bar{x})\right] \bar{s} y^T}{y^T y} \\ &\quad - \frac{F'(\bar{x})^+ \left[y - F'(\bar{x}) \bar{s}\right] y^T}{y^T y} - \left[F'(\bar{x}_+)^+ - F'(\bar{x})^+\right]. \end{aligned}$$

The last term on the right-hand side is $O(|\bar{s}|^p)$, and the next-to-last term is also $O(|\bar{s}|^p)$ for $\bar{s}$ in the range of K and K near $F'(\bar{x})^+$. However, the second term, which is in $\mathbf{N}(y)^\perp$, can only be bounded by a constant times $\|K - F'(\bar{x})^+\|$. Thus there appears to be no hope of realizing a bounded deterioration inequality or an appropriate counterpart of (2.20) for this update.

In formulating our analogues of Algorithm 1.3, we use a *choice rule* as in [14] and [6] for determining admissible right-hand sides of secant equations. Such a rule is given as a function χ which for each pair $\bar{x}$, $\bar{x}_+ \in \Omega$ determines a set $\chi(\bar{x}, \bar{x}_+) \subseteq \mathbf{R}^n$ in which admissible right-hand sides lie. In most cases of practical interest, there is no reason for making anything other than the traditional choice, corresponding to $\chi(\bar{x}, \bar{x}_+) = \{F(\bar{x}_+) - F(\bar{x})\}$. However, in some important contexts this may not be the preferred choice, and in some instances it may not be admissible in the analysis which follows. There is an extensive discussion in [14,§3] of choice rules and the conditions they must satisfy in the case of an equal number of equations and unknowns. That discussion is valid here with only minor appropriate changes, and we refer the reader to it.

Our first analogue of Algorithm 1.3 uses direct least-change secant updates.

ALGORITHM 2.4. *Given $\bar{x}_0 \in \mathbf{R}^{\bar{n}}$ and $B_0 \in \mathbf{R}^{n \times \bar{n}}$, determine for $k = 0, 1, \ldots,$*

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k - B_k^+ F(\bar{x}_k), \\ y_k &\in \chi(\bar{x}_k, \bar{x}_{k+1}), \\ B_{k+1} &= (B_k)_+, \end{aligned}$$

where $(B_k)_+$ is the least-change secant update of B_k in $\mathbf{A}$ with respect to $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k$, y_k, and the norm $\|\cdot\|$.

In Algorithm 2.4 and in the other algorithms below, we have not allowed the options of not updating at an iteration or of taking $B_k = C(\bar{x}_k) + A_k$, where $C(\bar{x}_k)$ is a "computed part" of $F'(\bar{x}_k)$ and A_k is an "approximated" part maintained by updating. Although these options are often a very important part of practically effective algorithms, we have omitted them here to simplify the exposition. It would be trivial to modify the analysis below to allow the option of not updating; it would be straightforward and not difficult to modify it to allow a "computed part" of $F'(\bar{x}_k)$ in each B_k. Under such modifications, the results below would still be valid with only minor appropriate changes.

Our local convergence analysis for Algorithm 2.4 is given in Theorems 2.5 and 2.6 below. Theorem 2.5 addresses the local q-linear convergence of the algorithm; Theorem 2.6 draws more refined conclusions about the asymptotic speed of convergence and, in particular, gives conditions under which the convergence is q-superlinear. We comment further on these theorems and the conditions in them following the proofs.

Our notation and terminology used in association with q-linear and q-superlinear convergence is that of Ortega and Rheinboldt [22, p.281]: If $\{z_k\}_{k=0,1,\dots}$ is a sequence converging to z_* in a finite-dimensional vector space with norm $|\cdot|$, then $Q_1\{z_k\}$, the *linear q-factor of* $\{z_k\}_{k=0,1,\dots}$, is defined as

$$Q_1\{z_k\} = \begin{cases} 0, & \text{if } z_k = z_*,\ k \geq \text{some } k_0, \\ \overline{\lim}_{k\to\infty} |z_{k+1} - z_*|/|z_k - z_*|, & \text{if } z_k \neq z_*,\ k \geq \text{some } k_0, \\ +\infty, & \text{otherwise.} \end{cases}$$

One says that $\{z_k\}_{k=0,1,\dots}$ converges q-linearly to z_* in the norm $|\cdot|$ if and only if $Q_1\{z_k\} < 1$ and that $\{z_k\}_{k=0,1,\dots}$ converges q-superlinearly to z_* if and only if $Q_1\{z_k\} = 0$. Note that in a finite-dimensional vector space q-superlinear convergence holds in one norm if and only if it holds in every other norm.

THEOREM 2.5. *Let F satisfy Hypothesis 1.2 and suppose Ω_η is given by (1.1) for some $\eta > 0$. Assume that χ has the property with $\mathbf{A}$ that there exists an $\alpha \geq 0$ such that for any $\bar{x}$, $\bar{x}_+ \in \Omega$ and any $y \in \chi(\bar{x}, \bar{x}_+)$, one has*

$$\left\| P^{\perp}_{\mathbf{S}\cap\mathbf{N}(\bar{s})}[G - P_{\mathbf{A}}F'(\bar{x})] \right\| \leq \alpha |\bar{s}|^p \tag{2.11}$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(y, \bar{s}))$, where $\bar{s} = \bar{x}_+ - \bar{x}$. Then for any $r \in (0,1)$ and $\mu' > \mu$, there are $\epsilon > 0$ and $\delta > 0$ such that if $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ satisfy $|F(\bar{x}_0)| < \epsilon$ and $|B_0 - F'(\bar{x}_0)| < \delta$, then the iterates $\{\bar{x}_k\}_{k=0,1,\dots}$ determined by Algorithm 2.4 are well-defined and converge q-linearly to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$ with

$$|\bar{x}_{k+1} - \bar{x}_*| \leq r|\bar{x}_k - \bar{x}_*|, \qquad k = 0, 1, \dots, \tag{2.12}$$

and with $\{|B_k^+|\}_{k=0,1,\dots}$ uniformly bounded by μ'. Also, $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\dots}$ is uniformly small.

Proof. We define an *update function* U on $\Omega \times \Omega \times \mathbf{A}$ (see the Appendix) by

$$U(\bar{x}, \bar{x}_+, B) = \{B_+ : y \in \chi(\bar{x}, \bar{x}_+)\},$$

where B_+ is the least-change secant update of B in $\mathbf{A}$ with respect to $\bar{s} = \bar{x}_+ - \bar{x}$, $y \in \chi(\bar{x}, \bar{x}_+)$, and the norm $\|\cdot\|$. We show below that Hypothesis A.2 of the Appendix holds for this update function. The theorem then follows from Theorem A.4 of the Appendix.

Since $\|B_+ - F'(\bar{x}_+)\| \le \|B_+ - F'(\bar{x})\| + O(|\bar{s}|^p)$ by Hypothesis 1.2, it suffices to show that there are nonnegative constants α_1 and α_2 such that for $(\bar{x}, \bar{x}_+, B) \in \Omega \times \Omega \times \mathbf{A}$, every $B_+ \in U(\bar{x}, \bar{x}_+, B)$ satisfies

$$\|B_+ - F'(\bar{x})\| \le (1 + \alpha_1 |\bar{s}|^p) \|B - F'(\bar{x})\| + \alpha_2 |\bar{s}|^p. \tag{2.13}$$

From (2.10) and (2.11), one has

$$\begin{aligned}
\|B_+ - P_{\mathbf{A}} F'(\bar{x})\| &\le \left\|P_{\mathbf{S} \cap \mathbf{N}(\bar{s})}[B - P_{\mathbf{A}} F'(\bar{x})]\right\| + \left\|P^{\perp}_{\mathbf{S} \cap \mathbf{N}(\bar{s})}[G - P_{\mathbf{A}} F'(\bar{x})]\right\| \\
&\le \|B - P_{\mathbf{A}} F'(\bar{x})\| + \alpha |\bar{s}|^p,
\end{aligned}$$

where $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(y, \bar{s}))$. Since B, $B_+ \in \mathbf{A}$, it follows that

$$\begin{aligned}
\|B_+ - F'(\bar{x})\|^2 &= \|B_+ - P_{\mathbf{A}} F'(\bar{x})\|^2 + \left\|P^{\perp}_{\mathbf{A}} F'(\bar{x})\right\|^2 \\
&\le \|B - P_{\mathbf{A}} F'(\bar{x})\|^2 + 2 \|B - P_{\mathbf{A}} F'(\bar{x})\| \cdot \alpha |\bar{s}|^p \\
&\qquad + \alpha^2 |\bar{s}|^{2p} + \left\|P^{\perp}_{\mathbf{A}} F'(\bar{x})\right\|^2 \\
&\le \left(\|B - F'(\bar{x})\| + \alpha |\bar{s}|^p\right)^2.
\end{aligned}$$

Then (2.13) holds with $\alpha_1 = 0$ and $\alpha_2 = \alpha$, and the proof is complete. □

Remark. The dependence of ϵ and δ is too ungainly to state in Theorem 2.5, but for the record and for the analysis in §3, we note that it follows from the proof of Theorem 2.5 and from the Remark after the proof of Theorem A.4 in the Appendix that ϵ and δ depend only on r, μ', γ, p, η, α in (2.11), and a bound on $|F'(\bar{x})|$ for $\bar{x} \in \Omega_{\eta/2}$ (as well as on the norms $|\cdot|$ and $\|\cdot\|$).

THEOREM 2.6. *Suppose that the hypotheses of Theorem 2.5 hold and that* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *is a sequence generated by Algorithm 2.4 which converges q-linearly to* $\bar{x}_* \in \Omega$ *with (2.12) satisfied for some* $r \in (0, 1)$ *and with* $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k \neq 0$ *for all* k. *Then for* $B_* = P_{\mathbf{A}} F'(\bar{x}_*)$, *one has*

$$\lim_{k \to \infty} \frac{|[B_* - F'(\bar{x}_*)](\bar{x}_k - \bar{x}_*) - B_*(\bar{x}_{k+1} - \bar{x}_*)|}{|\bar{x}_k - \bar{x}_*|} = 0, \tag{2.14}$$

which implies

$$\overline{\lim_{k\to\infty}} \frac{\left|B_*{}^+ B_*(\bar{x}_{k+1} - \bar{x}_*)\right|}{|\bar{x}_k - \bar{x}_*|} = \overline{\lim_{k\to\infty}} \frac{\left|B_*{}^+ \left[B_* - F'(\bar{x}_*)\right](\bar{x}_k - \bar{x}_*)\right|}{|\bar{x}_k - \bar{x}_*|} \le \left|B_*{}^+ \left[B_* - F'(\bar{x}_*)\right]\right|. \tag{2.15}$$

It follows that if $\left\{|B_k^+|\right\}_{k=0,1,\ldots}$ *is uniformly bounded by* μ' *and if* $r < (1 + \mu'|B_*|)^{-1}$, *then*

$$\lim_{k\to\infty} \frac{|\bar{x}_{k+1} - \bar{x}_*|}{|\bar{x}_k - \bar{x}_*|} \le \frac{\mu'|B_*|}{1 - r(1 + \mu'|B_*|)} \left|B_*{}^+ \left[B_* - F'(\bar{x}_*)\right]\right|. \tag{2.16}$$

In particular, if $F'(\bar{x}_*) \in \mathbf{A}$ *as well, then* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *converges q-superlinearly to* $\bar{x}_*$.

Proof. It suffices to show that

$$\lim_{k\to\infty} \frac{|(B_k - B_*)\bar{s}_k|}{|\bar{s}_k|} = 0. \tag{2.17}$$

With (2.17), the theorem follows from Theorem A.5 and Proposition A.6 of the Appendix. One can show as in Lemma 3.4 of [14] that (2.17) is equivalent to

$$\lim_{k\to\infty} \left\| P^{\perp}_{\mathbf{N}(\bar{s}_k)}(B_k - B_*) \right\| = 0, \tag{2.18}$$

and so we establish (2.18).

We note that (2.10) gives

$$\begin{aligned} \|B_{k+1} - B_*\| &\le \|B_{k+1} - P_{\mathbf{A}}F'(\bar{x}_k)\| + \|P_{\mathbf{A}}F'(\bar{x}_k) - B_*\| \\ &\le \left\|P_{\mathbf{S}\cap\mathbf{N}(\bar{s}_k)}[B_k - P_{\mathbf{A}}F'(\bar{x}_k)]\right\| + \left\|P^{\perp}_{\mathbf{S}\cap\mathbf{N}(\bar{s}_k)}[G_k - P_{\mathbf{A}}F'(\bar{x}_k)]\right\| \\ &\quad + \|P_{\mathbf{A}}F'(\bar{x}_k) - B_*\| \\ &\le \left\|P_{\mathbf{S}\cap\mathbf{N}(\bar{s}_k)}(B_k - B_*)\right\| + \left\|P^{\perp}_{\mathbf{S}\cap\mathbf{N}(\bar{s}_k)}[G_k - P_{\mathbf{A}}F'(\bar{x}_k)]\right\| \\ &\quad + 2\,\|P_{\mathbf{A}}F'(\bar{x}_k) - B_*\| \end{aligned} \tag{2.19}$$

for $G_k \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(y_k, \bar{s}_k))$. We set $\sigma_k = \max\{|\bar{x}_k - \bar{x}_*|, |\bar{x}_{k+1} - \bar{x}_*|\}$ and note that $|\bar{s}_k| \le 2\sigma_k$. From (2.19), (2.11), Hypothesis 1.2, and the fact that $P_{\mathbf{S}\cap\mathbf{N}(\bar{s}_k)} = P_{\mathbf{S}\cap\mathbf{N}(\bar{s}_k)} \cdot P_{\mathbf{N}(\bar{s}_k)}$, one obtains

$$\|B_{k+1} - B_*\| \le \left\|P_{\mathbf{N}(\bar{s}_k)}(B_k - B_*)\right\| + \kappa\sigma_k^p \tag{2.20}$$

for $\kappa = 2^p\alpha + 2\gamma$. This is analogous to inequality (3.15) in the proof of Theorem 3.3, pages 965–966, of [14]. One proceeds in the manner of that proof to establish (2.18). □

The main import of Theorems 2.5 and 2.6 is the following: If the various hypotheses are satisfied, then the iterates generated by Algorithm 2.4 converge q-superlinearly to a solution $\bar{x}_*$ of Problem 1.1 provided $|F(\bar{x}_0)|$ and $|B_0 - F'(\bar{x}_0)|$ are sufficiently small for $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ and provided $F'(\bar{x}_*) \in \mathbf{A}$. If $F'(\bar{x}_*) \notin \mathbf{A}$, then (2.16) provides a statement of asymptotic q-linear convergence analogous to that of Theorem 3.3 of [14]. This statement is not as satisfying as that of [14]: For one thing, the factor $(\mu'|B_*|)/[1 - r(1 + \mu'|B_*|)]$ precludes this statement from reducing to that of Theorem 3.3 of [14] when $\bar{n} = n$; however, if $\bar{n} = n$, then (2.15) becomes the statement of Theorem 3.3 of [14]. For another, it might not be possible to satisfy $r < (1 + \mu'|B_*|)^{-1}$ if $F'(\bar{x}_*) \notin \mathbf{A}$. However, we note that if $\bar{x}_* \in \Omega$ is any point such that $F(\bar{x}_*) = 0$ and $F'(\bar{x}_*) \in \mathbf{A}$, then $\bar{x}_* \in \Omega_\eta$ for some $\eta > 0$, and one can find $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ for which $|F(\bar{x}_0)|$ and $|B_0 - F'(\bar{x}_0)|$ are arbitrarily small. In this case then, Theorems 2.5 and 2.6 imply that there are $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ for which the iterates produced by Algorithm 2.4 exhibit at least arbitrarily fast q-linear convergence to a zero of F, not necessarily $\bar{x}_*$.

If $F'(\bar{x}_*) \in \mathbf{A}$ at every $\bar{x}_* \in \Omega$ for which $F(\bar{x}_*) = 0$, then one can draw stronger conclusions from Theorems 2.5 and 2.6, which we summarize in Corollary 2.7 below. Under the hypotheses of this corollary, if Problem 1.1 has a solution in Ω, then there exist $\bar{x}_0 \in \Omega$ and $B_0 \in \mathbf{A}$ for which the iterates generated by Algorithm 2.4 converge q-superlinearly to some solution in Ω.

COROLLARY 2.7. *Let F satisfy Hypothesis 1.2, let Ω_η be given by (1.1) for some $\eta > 0$, and suppose $F'(\bar{x}_*) \in \mathbf{A}$ for all $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$. Assume that χ has the property with $\mathbf{A}$ that there exists an $\alpha \geq 0$ such that for any $\bar{x}$, $\bar{x}_+ \in \Omega$ and any $y \in \chi(\bar{x}, \bar{x}_+)$, one has*

$$\left\| P^{\perp}_{\mathbf{S} \cap \mathbf{N}(\bar{s})}[G - P_{\mathbf{A}} F'(\bar{x})] \right\| \leq \alpha |\bar{s}|^p \tag{2.21}$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(y, \bar{s}))$, where $\bar{s} = \bar{x}_+ - \bar{x}$. Then there are $\epsilon > 0$ and $\delta > 0$ such that if $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ satisfy $|F(\bar{x}_0)| < \epsilon$ and $|B_0 - F'(\bar{x}_0)| < \delta$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm 2.4 are well-defined and converge q-superlinearly to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$. Also, $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ is uniformly small and $\{|B_k^+|\}_{k=0,1,\ldots}$ is uniformly bounded with a bound near μ.

In the most important case of most general interest, one has $F'(\bar{x}) \in \mathbf{A}$ for all $\bar{x} \in \Omega$. In this case, for any $\bar{x}_0 \in \Omega$, one can always make $|B_0 - F'(\bar{x}_0)|$ arbitrarily small for $B_0 \in \mathbf{A}$. Then in this case, under the hypotheses of Corollary 2.7, one can always obtain q-superlinear convergence of the iterates of Algorithm 2.4 to a solution of Problem 1.1 whenever $|F(\bar{x}_0)|$ is sufficiently small by ensuring in addition that $|B_0 - F'(\bar{x}_0)|$ is sufficiently small for $B_0 \in \mathbf{A}$. It is important to note that when $F'(\bar{x}) \in \mathbf{A}$ for all $\bar{x} \in \Omega$, the choice rule $\chi(\bar{x}, \bar{x}_+) = \{F(\bar{x}_+) - F(\bar{x})\}$ satisfies (2.11) and (2.21) under Hypothesis 1.2. Indeed, $F'(\bar{x}) = P_{\mathbf{A}} F'(\bar{x})$ in this case, and if

$$y = F(\bar{x}_+) - F(\bar{x}) = \left\{ \int_0^1 F'(\bar{x}(t))\, dt \right\} \bar{s}$$

for $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{x}(t) = \bar{x} + t\bar{s}$, then $G \equiv \int_0^1 F'(\bar{x}(t))\,dt \in \mathbf{A} \cap \mathbf{Q}(y, \bar{s}) = \mathbf{M}(\mathbf{A}, \mathbf{Q}(y, \bar{s}))$ satisfies

$$\left\| P^{\perp}_{\mathbf{S} \cap \mathbf{N}(\bar{s})}[G - F'(\bar{x})] \right\| \leq \left\| \int_0^1 [F'(\bar{x}(t)) - F'(\bar{x})]\,dt \right\|$$
$$\leq \frac{\gamma}{1+p}|\bar{s}|^p$$

by Hypothesis 1.2. Since (2.21) holds for this G, it also holds for every element of $\mathbf{M}(\mathbf{A}, \mathbf{Q}(y, \bar{s}))$; see [14]. Thus we stress that under Hypothesis 1.2, *when* $F'(\bar{x}) \in \mathbf{A}$ *for all* $\bar{x} \in \Omega$, *the conclusions of Corollary 2.7 hold when one makes the traditional choice* $y_k = F(\bar{x}_{k+1}) - F(\bar{x}_k)$ *in Algorithm 2.4.* In particular, under Hypothesis 1.2, the conclusions of Corollary 2.7 hold with this choice of y_k in the following circumstances:

(i) when the update is the (first) Broyden update of [6];

(ii) when the update is the Powell symmetric Broyden update of [6], provided the appropriate subset of n columns of $F'(\bar{x})$ exhibits symmetry for all $\bar{x} \in \Omega$;

(iii) when the update is the sparse Broyden update of [6], provided $F'(\bar{x})$ has the appropriate pattern of sparsity for all $\bar{x} \in \Omega$;

(iv) when the update is the sparse symmetric update of [3], provided $F'(\bar{x})$ has the appropriate pattern of sparsity and the appropriate subset of n columns of $F'(\bar{x})$ exhibits symmetry for all $\bar{x} \in \Omega$.

Our second analogue of Algorithm 1.3 is Algorithm 2.8 below which uses inverse least-change secant updates. The local convergence analysis for this algorithm is given in Theorems 2.9 and 2.10 and Corollary 2.11 which follow. We omit the proofs of these results, which are similar to those of Theorems 2.5 and 2.6 and Corollary 2.7 but rely on the local convergence analysis for Algorithm A.7 of the Appendix instead of that for Algorithm A.1.

ALGORITHM 2.8. *Given* $\bar{x}_0 \in \mathbf{R}^{\bar{n}}$ *and* $B_0 \in \mathbf{R}^{n \times \bar{n}}$, *determine for* $k = 0, 1, \ldots,$

$$\begin{aligned}
\bar{x}_{k+1} &= \bar{x}_k - B_k^+ F(\bar{x}_k),\\
y_k &\in \chi(\bar{x}_k, \bar{x}_{k+1}),\\
K_k &= \left[\mathcal{B}_k^{-1}, -\mathcal{B}_k^{-1}\mathcal{C}_k\right], \ \textit{where } B_k = [\mathcal{B}_k, \mathcal{C}_k],\\
K_{k+1} &= (K_k)_+,\\
B_{k+1} &= \left[\mathcal{K}_{k+1}^{-1}, -\mathcal{K}_{k+1}^{-1}\mathcal{L}_{k+1}\right], \ \textit{where } K_{k+1} = [\mathcal{K}_{k+1}, \mathcal{L}_{k+1}],
\end{aligned}$$

where $(K_k)_+$ *is the least-change inverse secant update of* K_k *in* $\mathbf{A}$ *with respect to* $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k$, y_k, *and the norm* $\|\cdot\|$.

THEOREM 2.9. *Let* F *satisfy Hypothesis 1.2, suppose* $F_x(\bar{x})$ *is nonsingular for all* $\bar{x} \in \Omega$, *and suppose* Ω_η *is given by (1.1) for some* $\eta > 0$. *Assume that* χ *has the property with* $\mathbf{A}$ *that there exists an* $\alpha \geq 0$ *such that for any* $\bar{x} = (x, \lambda)$, $\bar{x}_+ = (x_+, \lambda_+) \in \Omega$ *and any* $y \in \chi(\bar{x}, \bar{x}_+)$, *one has*

$$\left\| P^{\perp}_{\mathbf{S} \cap \mathbf{N}(\bar{y})} \left\{ G - P_{\mathbf{A}}\left[F_x(\bar{x})^{-1}, -F_x(\bar{x})^{-1}F_\lambda(\bar{x})\right] \right\} \right\| \leq \alpha |\bar{s}|^p \tag{2.22}$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(s, \bar{y}))$, *where* $\bar{s} = \bar{x}_+ - \bar{x}$, $s = x_+ - x$, *and* $\bar{y} = (y, \lambda_+ - \lambda)$. *Then for any* $r \in (0,1)$ *and* $\mu' > \mu$, *there are* $\epsilon > 0$ *and* $\delta > 0$ *such that if* $\bar{x}_0 \in \Omega_\eta$ *and* $B_0 = [\mathcal{B}_0, \mathcal{C}_0]$ *with* $[\mathcal{B}_0^{-1}, -\mathcal{B}_0^{-1}\mathcal{C}_0] \in \mathbf{A}$ *satisfy* $|F(\bar{x}_0)| < \epsilon$ *and* $|B_0 - F'(\bar{x}_0)| < \delta$, *then the iterates* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *determined by Algorithm 2.8 are well-defined and converge q-linearly to a point* $\bar{x}_* \in \Omega$ *such that* $F(\bar{x}_*) = 0$ *with*

$$(2.23) \qquad |\bar{x}_{k+1} - \bar{x}_*| \le r|\bar{x}_k - \bar{x}_*|, \qquad k = 0, 1, \ldots,$$

and with $\{|B_k^+|\}_{k=0,1,\ldots}$ *uniformly bounded by* μ'. *Also,* $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ *is uniformly small.*

THEOREM 2.10. *Suppose that the hypotheses of Theorem 2.9 hold and that* $\{\bar{x}_k = (x_k, \lambda_k)\}_{k=0,1,\ldots}$ *is a sequence generated by Algorithm 2.8 which converges q-linearly to* $\bar{x}_* \in \Omega$ *with (2.23) satisfied for some* $r \in (0,1)$, *with* $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k \neq 0$ *for all* k, *and with* $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ *uniformly small. Set*

$$K_* = [\mathcal{K}_*, \mathcal{L}_*] = P_{\mathbf{A}}\left[F_x(\bar{x}_*)^{-1}, -F_x(\bar{x}_*)^{-1}F_\lambda(\bar{x}_*)\right],$$

and assume further that $\mathcal{K}_*$ *is invertible and that* $\{y_k\}_{k=0,1,\ldots}$ *satisfies* $|K_*\bar{y}_k - s_k| \le \alpha_k|\bar{s}_k|$ *for each* k, *where* $\bar{y}_k = (y_k, \lambda_{k+1} - \lambda_k)$, $s_k = x_{k+1} - x_k$, *and* $\lim_{k\to\infty} \alpha_k = 0$. *Then (2.14) and (2.15) hold with* $B_* = \left[\mathcal{K}_*^{-1}, -\mathcal{K}_*^{-1}\mathcal{L}_*\right]$. *It follows that if* $\{|B_k^+|\}_{k=0,1,\ldots}$ *is uniformly bounded by* μ' *and if* $r < (1 + \mu'|B_*|)^{-1}$, *then (2.16) holds with this* B_*. *In particular, if* $\left[F_x(\bar{x}_*)^{-1}, -F_x(\bar{x}_*)^{-1}F_\lambda(\bar{x}_*)\right] \in \mathbf{A}$ *as well, then* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *converges q-superlinearly to* $\bar{x}_*$.

COROLLARY 2.11. *Let* F *satisfy Hypothesis 1.2, suppose* $F_x(\bar{x})$ *is nonsingular for all* $\bar{x} \in \Omega$, *let* Ω_η *be given by (1.1) for some* $\eta > 0$, *and suppose* $\left[F_x(\bar{x}_*)^{-1}, -F_x(\bar{x}_*)^{-1}F_\lambda(\bar{x}_*)\right] \in \mathbf{A}$ *for all* $\bar{x}_* \in \Omega$ *such that* $F(\bar{x}_*) = 0$. *Assume that* χ *has the property with* $\mathbf{A}$ *that there exists an* $\alpha \ge 0$ *such that for any* $\bar{x} = (x, \lambda)$, $\bar{x}_+ = (x_+, \lambda_+) \in \Omega$ *and any* $y \in \chi(\bar{x}, \bar{x}_+)$, *one has*

$$\left\| P^\perp_{\mathbf{S}\cap\mathbf{N}(\bar{y})}\left\{G - P_{\mathbf{A}}\left[F_x(\bar{x})^{-1}, -F_x(\bar{x})^{-1}F_\lambda(\bar{x})\right]\right\}\right\| \le \alpha|\bar{s}|^p$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(s, \bar{y}))$, *where* $\bar{s} = \bar{x}_+ - \bar{x}$, $s = x_+ - x$, *and* $\bar{y} = (y, \lambda_+ - \lambda)$. *Then there are* $\epsilon > 0$ *and* $\delta > 0$ *such that if* $\bar{x}_0 \in \Omega_\eta$ *and* $B_0 = [\mathcal{B}_0, \mathcal{C}_0]$ *with* $[\mathcal{B}_0^{-1}, -\mathcal{B}_0^{-1}\mathcal{C}_0] \in \mathbf{A}$ *satisfy* $|F(\bar{x}_0)| < \epsilon$ *and* $|B_0 - F'(\bar{x}_0)| < \delta$, *then the iterates* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *determined by Algorithm 2.8 are well-defined and converge q-superlinearly to a point* $\bar{x}_* \in \Omega$ *such that* $F(\bar{x}_*) = 0$. *Also,* $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ *is uniformly small and* $\{|B_k^+|\}_{k=0,1,\ldots}$ *is uniformly bounded with a bound near* μ.

Remarks similar to those following Theorem 2.6 and Corollary 2.7 are valid in the context of Algorithm 2.8, Theorems 2.9 and 2.10, and Corollary 2.11. We note explicitly only that under Hypothesis 1.2 and the assumption that $F_x(\bar{x})$ is nonsingular for all $\bar{x} \in \Omega$, *if* $\left[F_x(\bar{x})^{-1}, -F_x(\bar{x})^{-1}F_\lambda(\bar{x})\right] \in \mathbf{A}$ *for all* $\bar{x} \in \Omega$, *then the conclusions of Corollary 2.11 hold when one makes the traditional choice* $y_k = F(\bar{x}_{k+1}) - F(\bar{x}_k)$ *in Algorithm 2.8.* In particular, under Hypothesis 1.2 and the assumption that $F_x(\bar{x})$ is nonsingular for all $\bar{x} \in \Omega$, the conclusions of Corollary 2.11 hold with this choice of y_k in the following circumstances:

(i) when the update is the second Broyden update of [6];

(ii) when the update is the Greenstadt update of [6], provided $F_x(\bar{x})$ is symmetric for all $\bar{x} \in \Omega$.

3. The augmented Jacobian algorithm. We now consider Algorithm 1.4 and its analogues which use least-change secant and inverse secant updates to approximate F'. Throughout this section we suppose $V \in \mathbf{R}^{m \times \bar{n}}$ is given, and instead of Hypothesis 1.2 we use the following

HYPOTHESIS 3.1. *F is differentiable and* $\begin{bmatrix} F'(\bar{x}) \\ V \end{bmatrix}$ *is nonsingular in an open convex set Ω, and the following hold:*

(i) there exist $\gamma \geq 0$ and $p \in (0,1]$ such that $|F'(\bar{y}) - F'(\bar{x})| \leq \gamma\, |\bar{y} - \bar{x}|^p$ for all $\bar{x}$, $\bar{y} \in \Omega$;

(ii) there is a constant $\bar{\mu}$ for which $\left| \begin{bmatrix} F'(\bar{x}) \\ V \end{bmatrix}^{-1} \right| \leq \bar{\mu}$ *for all $\bar{x} \in \Omega$.*

We note that Hypothesis 3.1 on F and V implies Hypothesis 1.2 on F with $\mu \leq \bar{\mu}$.

Throughout the following, for given $\bar{x}_0 \in \Omega$ we define

$$\bar{F}(\bar{x}) \equiv \begin{pmatrix} F(\bar{x}) \\ V(\bar{x} - \bar{x}_0) \end{pmatrix} \tag{3.1}$$

for $\bar{x} \in \Omega$. Of course $\bar{F}$ depends on $\bar{x}_0$, but it is convenient to suppress this dependence in the notation. From

$$\bar{F}'(\bar{x}) = \begin{bmatrix} F'(\bar{x}) \\ V \end{bmatrix}, \tag{3.2}$$

one sees that Hypothesis 3.1 on F and V implies the following:

(i) there exists $\bar{\gamma} \geq 0$ depending only on γ such that

$$\left|\bar{F}'(\bar{y}) - \bar{F}'(\bar{x})\right| \leq \bar{\gamma}\, |\bar{y} - \bar{x}|^p \tag{3.3}$$

for all $\bar{x}$, $\bar{y} \in \Omega$;

(ii) $\bar{F}'(\bar{x})$ is nonsingular and $|\bar{F}'(\bar{x})^{-1}| \leq \bar{\mu}$ for all $\bar{x} \in \Omega$.

Thus Hypothesis 3.1 on F and V implies Hypothesis 1.2 on $\bar{F}$ with n, γ, and μ replaced by $\bar{n}$, $\bar{\gamma}$, and $\bar{\mu}$, respectively.

Our approach to the local convergence analyses of Algorithm 1.4 and its analogue below which uses least-change secant updates is to observe that these algorithms are respectively equivalent to Algorithms 1.3 and 2.4 applied to $\bar{F}$ in the $\bar{n} = n$ case. (Related observations are made for special cases, e. g., in [15] and [31].) The desired local convergence results are then obtained from the results in §2. An apparent difficulty with this approach is that $\bar{F}$ depends on $\bar{x}_0$, and so presumably the ϵ's and δ's of the theorems must also depend on $\bar{x}_0$, which would be unacceptable. However, one sees below that this difficulty is illusory because $\bar{F}'$ is independent of $\bar{x}_0$; see (3.2). We begin with a local convergence theorem for Algorithm 1.4 which is the counterpart of Theorem 2.1 for Algorithm 1.3.

THEOREM 3.2. *Let F and V satisfy Hypothesis 3.1 and suppose Ω_η is given by (1.1) for some $\eta > 0$. Then there is an $\epsilon > 0$ depending only on γ, p, $\bar{\mu}$, and η such that if $\bar{x}_0 \in \Omega_\eta$ and $|F(\bar{x}_0)| < \epsilon$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm 1.4 are well-defined and converge to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$. Furthermore, there is a constant β for which*

$$|\bar{x}_{k+1} - \bar{x}_*| \leq \beta |\bar{x}_k - \bar{x}_*|^{1+p}, \qquad k = 0, 1, \ldots. \tag{3.4}$$

Proof. Since Hypothesis 3.1 on F and V implies Hypothesis 1.2 on $\bar{F}$ given by (3.1) with n, γ, and μ replaced by $\bar{n}$, $\bar{\gamma}$ of (3.3), and $\bar{\mu}$, respectively, and since $\bar{\gamma}$ depends only on γ, it follows from Theorem 2.1 that there is an $\bar{\epsilon} > 0$ depending only on γ, p, $\bar{\mu}$, and η such that if $\bar{x}_0 \in \Omega_\eta$ and $|\bar{F}(\bar{x}_0)| < \bar{\epsilon}$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm 1.3 applied to $\bar{F}$ are well-defined and converge to a point $\bar{x}_* \in \Omega$ such that $\bar{F}(\bar{x}_*) = 0$, which implies $F(\bar{x}_*) = 0$, with (3.4) holding for some β. But for given $\bar{x}_0$, it is easy to see that Algorithm 1.3 applied to $\bar{F}$ is equivalent to Algorithm 1.4 applied to F. Letting $\epsilon > 0$ be such that $|\bar{F}(\bar{x}_0)| < \bar{\epsilon}$ whenever $|F(\bar{x}_0)| < \epsilon$ completes the proof. $\square$

As in §2, we assume throughout the following that $\mathbf{A} \subseteq \mathbf{R}^{n \times \bar{n}}$ is an affine subspace in which updated matrices are to lie and that χ is a choice rule for determining admissible right-hand sides of secant equations. We formulate an analogue of Algorithm 1.4 which uses direct least-change secant updates as

ALGORITHM 3.3. *Given $\bar{x}_0 \in \mathbf{R}^{\bar{n}}$, $B_0 \in \mathbf{R}^{n \times \bar{n}}$, and $V \in \mathbf{R}^{m \times \bar{n}}$ such that $\begin{bmatrix} B_0 \\ V \end{bmatrix}$ is nonsingular, determine for $k = 0, 1, \ldots,$*

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k + \bar{s}_k, \text{ where } B_k \bar{s}_k = -F(\bar{x}_k) \text{ and } V\bar{s}_k = 0, \\ y_k &\in \chi(\bar{x}_k, \bar{x}_{k+1}), \\ B_{k+1} &= (B_k)_+, \end{aligned}$$

where $(B_k)_+$ is the least-change secant update of B_k in $\mathbf{A}$ with respect to $\bar{s}_k$, y_k, and the norm $\|\cdot\|$.

Theorems 3.4 and 3.5 and Corollary 3.6 below are counterparts for Algorithm 3.3 of Theorems 2.5 and 2.6 and Corollary 2.7 for Algorithm 2.4.

THEOREM 3.4. *Let F satisfy Hypothesis 3.1 and suppose Ω_η is given by (1.1) for some $\eta > 0$. Assume that χ has the property with* $\mathbf{A}$ *that there exists an $\alpha \geq 0$ such that for any $\bar{x}$, $\bar{x}_+ \in \Omega$ and any $y \in \chi(\bar{x}, \bar{x}_+)$, one has*

$$\left\| P^{\perp}_{\mathbf{S}\cap\mathbf{N}(\bar{s})}[G - P_{\mathbf{A}}F'(\bar{x})] \right\| \leq \alpha|\bar{s}|^p \tag{3.5}$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(y, \bar{s}))$, where $\bar{s} = \bar{x}_+ - \bar{x}$. Then for any $r \in (0,1)$ and $\mu' > \bar{\mu}$, there are $\epsilon > 0$ and $\delta > 0$ such that if $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ satisfy $|F(\bar{x}_0)| < \epsilon$ and $|B_0 - F'(\bar{x}_0)| < \delta$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm 3.3 are well-defined and converge q-linearly to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$ with

$$|\bar{x}_{k+1} - \bar{x}_*| \leq r|\bar{x}_k - \bar{x}_*|, \qquad k = 0, 1, \ldots, \tag{3.6}$$

and with $\left\{ \left| \begin{bmatrix} B_k \\ V \end{bmatrix}^{-1} \right| \right\}_{k=0,1,\ldots}$ uniformly bounded by μ'. Also, $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ is uniformly small.

Proof. Given $\bar{x}_0$, B_0, and V, one sees by an easy induction that with $\bar{F}$ given by (3.1), the iteration of Algorithm 3.3 is equivalent to

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k - \begin{bmatrix} B_k \\ V \end{bmatrix}^{-1} \bar{F}(\bar{x}_k), \\ y_k &\in \chi(\bar{x}_k, \bar{x}_{k+1}), \\ B_{k+1} &= (B_k)_+, \end{aligned} \tag{3.7}$$

where $(B_k)_+$ is the least-change secant update of B_k in $\mathbf{A}$ with respect to $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k$, y_k, and the norm $\|\cdot\|$. We define an affine subspace $\bar{\mathbf{A}} \subseteq \mathbf{R}^{\bar{n}\times\bar{n}}$ by

$$\bar{\mathbf{A}} = \left\{ \bar{M} = \begin{bmatrix} M \\ V \end{bmatrix} : M \in \mathbf{A} \right\}$$

and an inner-product norm on $\mathbf{R}^{\bar{n}\times\bar{n}}$ as follows: Letting $\|\cdot\|$ be any inner-product norm on $\mathbf{R}^{m\times\bar{n}}$, we define a norm $\|\cdot\|$ on $\mathbf{R}^{\bar{n}\times\bar{n}}$ by

$$\|\bar{M}\|^2 = \|M\|^2 + \|N\|^2$$

for $\bar{M} = \begin{bmatrix} M \\ N \end{bmatrix} \in \mathbf{R}^{\bar{n}\times\bar{n}}$, $M \in \mathbf{R}^{n\times\bar{n}}$, $N \in \mathbf{R}^{m\times\bar{n}}$. Then the iteration (3.7) is equivalent to

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k - \begin{bmatrix} B_k \\ V \end{bmatrix}^{-1} \bar{F}(\bar{x}_k), \\ y_k &\in \chi(\bar{x}_k, \bar{x}_{k+1}), \\ \begin{bmatrix} B_{k+1} \\ V \end{bmatrix} &= \left(\begin{bmatrix} B_k \\ V \end{bmatrix} \right)_+, \end{aligned} \tag{3.8}$$

where $\left(\begin{bmatrix} B_k \\ V \end{bmatrix}\right)_+$ is the least-change secant update of $\begin{bmatrix} B_k \\ V \end{bmatrix}$ in $\bar{\mathbf{A}}$ with respect to $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k$, $(y_k, 0)$, and the norm $\|\cdot\|$ on $\mathbf{R}^{\bar{n}\times\bar{n}}$. We note that (3.8) is just an instance of the iteration of Algorithm 2.4. We also note that it follows from (3.5) that for any $\bar{x}$, $\bar{x}_+ \in \Omega$ and any $y \in \chi(\bar{x}, \bar{x}_+)$, one has

$$\left\| P^{\perp}_{\bar{\mathbf{S}}\cap\mathbf{N}(\bar{s})}[\bar{G} - P_{\bar{\mathbf{A}}}\bar{F}'(\bar{x})] \right\| \leq \alpha|\bar{s}|^p$$

for every $\bar{G} \in \mathbf{M}(\bar{\mathbf{A}}, \mathbf{Q}((y,0),\bar{s}))$, where $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{\mathbf{S}}$ is the parallel subspace of $\bar{\mathbf{A}}$.

Since Hypothesis 3.1 on F and V implies Hypothesis 1.2 on $\bar{F}$ with n, γ, and μ replaced by $\bar{n}$, $\bar{\gamma}$ of (3.3) depending only on γ, and $\bar{\mu}$, respectively, it follows from the above observations and from Theorem 2.5 applied to iteration (3.8) that for any $r \in (0,1)$ and $\mu' > \bar{\mu}$, there are $\bar{\epsilon} > 0$ and $\bar{\delta} > 0$ such that if $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ satisfy $|\bar{F}(\bar{x}_0)| < \bar{\epsilon}$ and $\left|\begin{bmatrix} B_0 \\ V \end{bmatrix} - \bar{F}'(\bar{x}_0)\right| < \bar{\delta}$, then the iterates $\{\bar{x}_k\}_{k=0,1,\dots}$ determined by Algorithm 3.3 are well-defined and converge q-linearly to a point $\bar{x}_* \in \Omega$ such that $\bar{F}(\bar{x}_*) = 0$, which implies $F(\bar{x}_*) = 0$, with (3.6) holding, with $\left\{\left|\begin{bmatrix} B_k \\ V \end{bmatrix}^{-1}\right|\right\}_{k=0,1,\dots}$ uniformly bounded by μ', and with $\left\{\left|\begin{bmatrix} B_k \\ V \end{bmatrix} - \bar{F}'(\bar{x}_k)\right|\right\}_{k=0,1,\dots}$ uniformly small, which implies $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\dots}$ is uniformly small. One sees from the Remark after the proof of Theorem 2.5 that $\bar{\epsilon}$ and $\bar{\delta}$ do not depend on $\bar{x}_0$. If ϵ and δ are chosen such that $|\bar{F}(\bar{x}_0)| < \bar{\epsilon}$ whenever $|F(\bar{x}_0)| < \epsilon$ and $\left|\begin{bmatrix} B_0 \\ V \end{bmatrix} - \bar{F}'(\bar{x}_0)\right| < \bar{\delta}$ whenever $|B_0 - F'(\bar{x}_0)| < \delta$, then the theorem easily follows. □

THEOREM 3.5. *Suppose that the hypotheses of Theorem 3.4 hold and that $\{\bar{x}_k\}_{k=0,1,\dots}$ is a sequence generated by Algorithm 3.3 which converges q-linearly to $\bar{x}_* \in \Omega$ with (3.6) satisfied for some $r \in (0,1)$ and with $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k \neq 0$ for all k. Then for $B_* = P_{\mathbf{A}}F'(\bar{x}_*)$, one has*

$$\lim_{k\to\infty} \frac{\left|\begin{bmatrix} B_* - F'(\bar{x}_*) \\ 0 \end{bmatrix}(\bar{x}_k - \bar{x}_*) - \begin{bmatrix} B_* \\ V \end{bmatrix}(\bar{x}_{k+1} - \bar{x}_*)\right|}{|\bar{x}_k - \bar{x}_*|} = 0. \tag{3.9}$$

It follows that if $\begin{bmatrix} B_* \\ V \end{bmatrix}$ is invertible, then

$$\begin{aligned} \overline{\lim_{k\to\infty}} \frac{|(\bar{x}_{k+1} - \bar{x}_*)|}{|\bar{x}_k - \bar{x}_*|} &= \overline{\lim_{k\to\infty}} \frac{\left|\begin{bmatrix} B_* \\ V \end{bmatrix}^{-1}\begin{bmatrix} B_* - F'(\bar{x}_*) \\ 0 \end{bmatrix}(\bar{x}_k - \bar{x}_*)\right|}{|\bar{x}_k - \bar{x}_*|} \\ &\leq \left|\begin{bmatrix} B_* \\ V \end{bmatrix}^{-1}\begin{bmatrix} B_* - F'(\bar{x}_*) \\ 0 \end{bmatrix}\right|. \end{aligned} \tag{3.10}$$

In particular, if $F'(\bar{x}_*) \in \mathbf{A}$ as well, then $\{\bar{x}_k\}_{k=0,1,\ldots}$ converges q-superlinearly to $\bar{x}_*$.

Proof. It is observed in the proof of Theorem 3.4 that the iteration of Algorithm 3.3 is equivalent to the iteration (3.8) with $\bar{\mathbf{A}}$, the norm $\|\cdot\|$ on $\mathbf{R}^{\bar{n}\times\bar{n}}$, etc., as defined there, and that (3.8) is an instance of the iteration of Algorithm 2.4. Since $P_{\bar{\mathbf{A}}}\bar{F}'(\bar{x}_*) = \begin{bmatrix} B_* \\ V \end{bmatrix}$, the theorem follows easily from Theorem 2.6. $\square$

Remark. It is easy to see that $V(\bar{x}_{k+1}-\bar{x}_*) = 0$ for all k; therefore, (3.9) implies (2.14) and (2.15). If $\left\{\left|\begin{bmatrix} B_k \\ V \end{bmatrix}^{-1}\right|\right\}_{k=0,1,\ldots}$ is uniformly bounded by μ', then so is $\{|B_k^+|\}_{k=0,1,\ldots}$; and if $r < (1+\mu'|B_*|)^{-1}$ as well, then (2.16) also follows.

COROLLARY 3.6. *Let F satisfy Hypothesis 3.1, let Ω_η be given by (1.1) for some $\eta > 0$, and suppose $F'(\bar{x}_*) \in \mathbf{A}$ for all $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$. Assume that χ has the property with $\mathbf{A}$ that there exists an $\alpha \geq 0$ such that for any $\bar{x}$, $\bar{x}_+ \in \Omega$ and any $y \in \chi(\bar{x}, \bar{x}_+)$, one has*

$$\left\|P^{\perp}_{\mathbf{S}\cap\mathbf{N}(\bar{s})}[G - P_{\mathbf{A}}F'(\bar{x})]\right\| \leq \alpha|\bar{s}|^p$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(y,\bar{s}))$, where $\bar{s} = \bar{x}_+ - \bar{x}$. Then there are $\epsilon > 0$ and $\delta > 0$ such that if $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ satisfy $|F(\bar{x}_0)| < \epsilon$ and $|B_0 - F'(\bar{x}_0)| < \delta$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm 3.3 are well-defined and converge q-superlinearly to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$. Also, $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ is uniformly small and $\left\{\left|\begin{bmatrix} B_k \\ V \end{bmatrix}^{-1}\right|\right\}_{k=0,1,\ldots}$ is uniformly bounded with a bound near $\bar{\mu}$.

Remarks similar to those following Theorem 2.6 and Corollary 2.7 are valid in the present context. We mention explicitly that under Hypothesis 3.1, *if $F'(\bar{x}) \in \mathbf{A}$ for all $\bar{x} \in \Omega$, then the conclusions of Corollary 3.6 hold when one makes the traditional choice $y_k = F(\bar{x}_{k+1}) - F(\bar{x}_k)$ in Algorithm 3.3.* In particular, under Hypothesis 3.1, the conclusions of Corollary 3.6 hold with this choice of y_k in the circumstances (i)–(iv) outlined following Corollary 2.7.

Our final algorithm is the following analogue of Algorithm 1.4 which uses least-change inverse secant updates.

ALGORITHM 3.7. *Given $\bar{x}_0 \in \mathbf{R}^{\bar{n}}$, $B_0 \in \mathbf{R}^{n\times\bar{n}}$, and $V \in \mathbf{R}^{m\times\bar{n}}$ such that $\begin{bmatrix} B_0 \\ V \end{bmatrix}$ is nonsingular, determine for $k = 0, 1, \ldots,$*

$$\begin{aligned}
&\bar{x}_{k+1} = \bar{x}_k + \bar{s}_k, \ \text{where } B_k\bar{s}_k = -F(\bar{x}_k) \text{ and } V\bar{s}_k = 0,\\
&\quad y_k \in \chi(\bar{x}_k, \bar{x}_{k+1}),\\
&\quad K_k = \left[\mathcal{B}_k^{-1}, -\mathcal{B}_k^{-1}\mathcal{C}_k\right], \ \text{where } B_k = [\mathcal{B}_k, \mathcal{C}_k],\\
&K_{k+1} = (K_k)_+,\\
&B_{k+1} = \left[\mathcal{K}_{k+1}^{-1}, -\mathcal{K}_{k+1}^{-1}\mathcal{L}_{k+1}\right], \ \text{where } K_{k+1} = [\mathcal{K}_{k+1}, \mathcal{L}_{k+1}],
\end{aligned}$$

and where $(K_k)_+$ *is the least-change inverse secant update of* K_k *in* $\mathbf{A}$ *with respect to* $\bar{s}_k$, y_k, *and the norm* $\|\cdot\|$.

Theorems 3.8 and 3.9 and Corollary 3.10 below are counterparts for Algorithm 3.7 of Theorems 2.9 and 2.10 and Corollary 2.11 for Algorithm 2.8. Unlike Algorithm 3.3, Algorithm 3.7 cannot be interpreted as a special case of its counterpart algorithm in §2, and so the results below cannot be obtained from the corresponding results in §2. However, the arguments leading to the results below closely parallel those used to establish their counterparts in §2, and we sketch these parallels in lieu of giving full proofs.

THEOREM 3.8. *Let* F *satisfy Hypothesis 3.1, suppose* $F_x(\bar{x})$ *is nonsingular for all* $\bar{x} \in \Omega$, *and suppose* Ω_η *is given by (1.1) for some* $\eta > 0$. *Assume that* χ *has the property with* $\mathbf{A}$ *that there exists an* $\alpha \geq 0$ *such that for any* $\bar{x} = (x, \lambda)$, $\bar{x}_+ = (x_+, \lambda_+) \in \Omega$ *and any* $y \in \chi(\bar{x}, \bar{x}_+)$, *one has*

$$\left\| P^{\perp}_{\mathrm{S} \cap \mathrm{N}(\bar{y})} \left\{ G - P_{\mathbf{A}} \left[F_x(\bar{x})^{-1}, -F_x(\bar{x})^{-1} F_\lambda(\bar{x}) \right] \right\} \right\| \leq \alpha |\bar{s}|^p \tag{3.11}$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(s, \bar{y}))$, *where* $\bar{s} = \bar{x}_+ - \bar{x}$, $s = x_+ - x$, *and* $\bar{y} = (y, \lambda_+ - \lambda)$. *Then for any* $r \in (0,1)$ *and* $\mu' > \bar{\mu}$, *there are* $\epsilon > 0$ *and* $\delta > 0$ *such that if* $\bar{x}_0 \in \Omega_\eta$ *and* $B_0 = [\mathcal{B}_0, \mathcal{C}_0]$ *with* $[\mathcal{B}_0^{-1}, -\mathcal{B}_0^{-1}\mathcal{C}_0] \in \mathbf{A}$ *satisfy* $|F(\bar{x}_0)| < \epsilon$ *and* $|B_0 - F'(\bar{x}_0)| < \delta$, *then the iterates* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *determined by Algorithm 3.7 are well-defined and converge q-linearly to a point* $\bar{x}_* \in \Omega$ *such that* $F(\bar{x}_*) = 0$ *with*

$$|\bar{x}_{k+1} - \bar{x}_*| \leq r |\bar{x}_k - \bar{x}_*|, \qquad k = 0, 1, \ldots, \tag{3.12}$$

and with $\left\{ \left\| \begin{bmatrix} B_k \\ V \end{bmatrix}^{-1} \right\| \right\}_{k=0,1,\ldots}$ *uniformly bounded by* μ'. *Also,* $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ *is uniformly small.*

Proof. For given $\bar{x}_0$, B_0, and V, one verifies as in the proof of Theorem 3.4 that with $\bar{F}$ given by (3.1), the iteration of Algorithm 3.7 is equivalent to

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k - \begin{bmatrix} B_k \\ V \end{bmatrix}^{-1} \bar{F}(\bar{x}_k), \\ y_k &\in \chi(\bar{x}_k, \bar{x}_{k+1}), \\ K_k &= \left[\mathcal{B}_k^{-1}, -\mathcal{B}_k^{-1}\mathcal{C}_k\right], \text{ where } B_k = [\mathcal{B}_k, \mathcal{C}_k], \\ K_{k+1} &= (K_k)_+, \\ B_{k+1} &= \left[\mathcal{K}_{k+1}^{-1}, -\mathcal{K}_{k+1}^{-1}\mathcal{L}_{k+1}\right], \text{ where } K_{k+1} = [\mathcal{K}_{k+1}, \mathcal{L}_{k+1}], \end{aligned} \tag{3.13}$$

and where $(K_k)_+$ is the least-change inverse secant update of K_k in $\mathbf{A}$ with respect to $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k$, y_k, and the norm $\|\cdot\|$. From (3.11) and arguments analogous to those in the proof of Theorem 2.5, one sees that the update in (3.13) has the bounded deterioration property of Hypothesis A.8. The theorem follows by applying to (3.13) a slight modification of the reasoning leading to Theorem A.9. □

THEOREM 3.9. *Suppose that the hypotheses of Theorem 3.8 hold and that* $\{\bar{x}_k = (x_k, \lambda_k)\}_{k=0,1,\ldots}$ *is a sequence generated by Algorithm 3.7 which converges q-linearly to* $\bar{x}_* \in \Omega$ *with (3.12) satisfied for some* $r \in (0,1)$, *with* $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k \neq 0$ *for all* k, *and with* $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ *uniformly small. Set*

$$K_* = [\mathcal{K}_*, \mathcal{L}_*] = P_{\mathbf{A}}\left[F_x(\bar{x}_*)^{-1}, -F_x(\bar{x}_*)^{-1}F_\lambda(\bar{x}_*)\right],$$

and assume further that $\mathcal{K}_*$ *is invertible and that* $\{y_k\}_{k=0,1,\ldots}$ *satisfies* $|K_*\bar{y}_k - s_k| \leq \alpha_k|\bar{s}_k|$ *for each* k, *where* $\bar{y}_k = (y_k, \lambda_{k+1} - \lambda_k)$, $s_k = x_{k+1} - x_k$, *and* $\lim_{k\to\infty}\alpha_k = 0$. *Then (3.9) holds with* $B_* = \left[\mathcal{K}_*^{-1}, -\mathcal{K}_*^{-1}\mathcal{L}_*\right]$. *It follows that if* $\begin{bmatrix} B_* \\ V \end{bmatrix}$ *is invertible, then (3.10) holds with this* B_*. *In particular, if* $\left[F_x(\bar{x}_*)^{-1}, -F_x(\bar{x}_*)^{-1}F_\lambda(\bar{x}_*)\right] \in \mathbf{A}$ *as well, then* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *converges q-superlinearly to* $\bar{x}_*$.

Proof. Using inverse-update analogues of the arguments in the proof of Theorem 2.6, one has that

$$\lim_{k\to\infty}\frac{|(K_k - K_*)\bar{y}_k|}{|\bar{y}_k|} = 0.$$

As in the proof of Theorem A.10, this implies

$$\lim_{k\to\infty}\frac{|(B_k - B_*)\bar{s}_k|}{|\bar{s}_k|} = 0.$$

With the arguments used in the proof of Theorem A.5, this in turn gives

$$\lim_{k\to\infty}\frac{|[B_* - F'(\bar{x}_*)]\,(\bar{x}_k - \bar{x}_*) - B_*(\bar{x}_{k+1} - \bar{x}_*)|}{|\bar{x}_k - \bar{x}_*|} = 0.$$

Since $V(\bar{x}_{k+1} - \bar{x}_*) = 0$ for all k (cf. the Remark after the proof of Theorem 3.5), one has (3.9), and the theorem follows. □

The Remark following the proof of Theorem 3.5 is relevant here as well.

COROLLARY 3.10. *Let* F *satisfy Hypothesis 3.1, suppose* $F_x(\bar{x})$ *is nonsingular for all* $\bar{x} \in \Omega$, *let* Ω_η *be given by (1.1) for some* $\eta > 0$, *and suppose* $\left[F_x(\bar{x}_*)^{-1}, -F_x(\bar{x}_*)^{-1}F_\lambda(\bar{x}_*)\right] \in \mathbf{A}$ *for all* $\bar{x}_* \in \Omega$ *such that* $F(\bar{x}_*) = 0$. *Assume that* χ *has the property with* $\mathbf{A}$ *that there exists an* $\alpha \geq 0$ *such that for any* $\bar{x} = (x, \lambda)$, $\bar{x}_+ = (x_+, \lambda_+) \in \Omega$ *and any* $y \in \chi(\bar{x}, \bar{x}_+)$, *one has*

$$\left\|P^{\perp}_{\mathsf{S}\cap\mathsf{N}(\bar{y})}\left\{G - P_{\mathbf{A}}\left[F_x(\bar{x})^{-1}, -F_x(\bar{x})^{-1}F_\lambda(\bar{x})\right]\right\}\right\| \leq \alpha|\bar{s}|^p$$

for every $G \in \mathbf{M}(\mathbf{A}, \mathbf{Q}(s, \bar{y}))$, *where* $\bar{s} = \bar{x}_+ - \bar{x}$, $s = x_+ - x$, *and* $\bar{y} = (y, \lambda_+ - \lambda)$. *Then there are* $\epsilon > 0$ *and* $\delta > 0$ *such that if* $\bar{x}_0 \in \Omega_\eta$ *and* $B_0 = [\mathcal{B}_0, \mathcal{C}_0]$ *with* $[\mathcal{B}_0^{-1}, -\mathcal{B}_0^{-1}\mathcal{C}_0] \in \mathbf{A}$ *satisfy* $|F(\bar{x}_0)| < \epsilon$ *and* $|B_0 - F'(\bar{x}_0)| < \delta$, *then the iterates* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *determined by*

Algorithm 3.7 are well-defined and converge q-superlinearly to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$. Also, $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ is uniformly small and $\left\{\left|\begin{bmatrix} B_k \\ V \end{bmatrix}^{-1}\right|\right\}_{k=0,1,\ldots}$ is uniformly bounded with a bound near $\bar{\mu}$.

Remarks similar to those following Theorem 2.6 and Corollary 2.7 are valid here. We note explicitly that under Hypothesis 3.1 and the assumption that $F_x(\bar{x})$ is nonsingular for all $\bar{x} \in \Omega$, *if* $\left[F_x(\bar{x})^{-1}, -F_x(\bar{x})^{-1}F_\lambda(\bar{x})\right] \in \mathbf{A}$ *for all* $\bar{x} \in \Omega$*, then the conclusions of Corollary 3.10 hold when one makes the traditional choice* $y_k = F(\bar{x}_{k+1}) - F(\bar{x}_k)$ *in Algorithm 3.7.* In particular, under Hypothesis 3.1 and the assumption that $F_x(\bar{x})$ is nonsingular for all $\bar{x} \in \Omega$, the conclusions of Corollary 3.10 hold with this choice of y_k in the circumstances (i)–(ii) outlined following Corollary 2.11.

4. Some numerical experiments. In this section we discuss some numerical experiments involving the methods of interest here. As indicated in the introduction, our purpose is not to offer a broad computational study but to give some indication of the performance of these methods in their simplest forms and to outline some basic issues associated with them. The only updates we consider are the first and second Broyden updates of [6], given respectively by

$$B_+ = B + \frac{(y - B\bar{s})\bar{s}^T}{\bar{s}^T\bar{s}} \tag{4.1}$$

and

$$B_+ = B + \frac{(y - B\bar{s})\left(y^TB + (0, t^T)\right)}{y^TB\bar{s} + t^Tt}, \tag{4.2}$$

where $\bar{s} = (s, t)$ for $s \in \mathbf{R}^n$ and $t \in \mathbf{R}^m$. In writing (4.2), we assume that the first n columns of B constitute a nonsingular matrix.

In our first set of experiments, we compared the performance of Algorithm 1.3, Algorithm 2.4 using update (4.1), and Algorithm 2.8 using update (4.2) on instances of Problem 1.1 involving simple scalar-valued functions of two variables. For each $F : \mathbf{R}^2 \to \mathbf{R}^1$ and $\bar{x}_0 \in \mathbf{R}^2$, we took $B_0 = F'(\bar{x}_0)$ in Algorithms 2.4 and 2.8. For perspective, we included in our comparisons a *chord method*, i. e., an iteration

$$\bar{x}_{k+1} = \bar{x}_k - B_0^+ F(\bar{x}_k)$$

with $B_0 = F'(\bar{x}_0)$. In all trials, we let each algorithm run until $|F(\bar{x}_k)| \leq 10^{-12}$. All runs were made in double precision on a Digital Equipment Corporation MicroVAX II running Ultrix and using the $f77$ Fortran compiler.

We first took $F(\bar{x}) = x_1 - 2x_2^3 + 9x_2^2 - 12x_2$, the zero curve of which is a cubic with turning points at $\bar{x} = (5, 1)$, $(4, 2)$. For starting points near this cubic, the performance of the algorithms was about what one might expect on the basis of experience with their counterparts for finding a root of a system with an equal number of equations and unknowns:

Method	Number of Iterations	Final Iterate
Algorithm 1.3	7	(4.864,.7997)
Algorithm 2.4	10	(4.929,.8531)
Algorithm 2.8	10	(4.927,.8516)
Chord Method	273	(4.929,.8531)

Table 1. Results for $F(\bar{x}) = x_1 - 2x_2^3 + 9x_2^2 - 12x_2$ with $\bar{x}_0 = (5, 0)$.

Algorithm 1.3 found a point on the curve in a reasonably small number of iterations, Algorithms 2.4 and 2.8 with the above updates required a few more (but usually the same number), and the chord method often needed many more to meet the very small residual tolerance. All methods found approximately the same point. The results given in Table 1 for $\bar{x}_0 = (5, 0)$ are typical.

For starting points farther away from the cubic, greater differences in the performance of the methods became evident. The results given in Table 2 for $\bar{x}_0 = (0, 5)$ are typical. A striking feature of these results is that Algorithm 2.8 with the second Broyden update (4.2) did considerably better than Algorithm 2.4 with the first Broyden update (4.1). As it happens, the iterates (which we do not show here) indicated that neither update performed very well on this problem in that each gave rise to occasional steps which led far away from the ultimate limit. However, in all our trials, Algorithm 2.8 with the update (4.2) always performed at least as well as Algorithm 2.4 with the update (4.1) and often did significantly better, as in the case shown here. This is in contrast to the $\bar{n} = n$ case, in which the second Broyden update is generally regarded as inferior to the first.

Method	Number of Iterations	Final Iterate
Algorithm 1.3	9	(1.226,.1112)
Algorithm 2.4	30	(.06936,.005806)
Algorithm 2.8	17	(4.711,1.355)
Chord Method	208	(.06936,.005806)

Table 2. Results for $F(\bar{x}) = x_1 - 2x_2^3 + 9x_2^2 - 12x_2$ with $\bar{x}_0 = (0, 5)$.

Another striking feature of the results in Table 2 is that the four methods yielded three markedly distinct points on the curve. In view of the distance of the starting point from the curve, the differences in these points is understandable; however, it might seem surprising that the iterates of Algorithm 2.4 with the update (4.1) and those of the chord method converged to the same point within numerical limits, an event also seen in Table 1. In fact, this is no accident, since at the kth iteration of Algorithm 2.4 with the update (4.1), one has in general that $\bar{s}_k = -B_k^+ F(\bar{x}_k)$ is in the range of B_k^T, and so (4.1) implies that the range of B_{k+1}^T is contained in that of B_k^T and by induction that of B_0^T. It follows that in general *Algorithm 2.4 with the first Broyden update (4.1) can be viewed as a special case of Algorithm 3.3* in which V is any matrix for which the range of V^T is the null space of B_0. It also follows that in general *the iterates produced by Algorithm 2.4 with the first Broyden update (4.1) and therefore their limit (if it exists) must lie in the affine subspace* $\bar{x}_0 + \text{range}\,\{B_0^T\}$. Related observations have been made by Georg [15] in the $\bar{n} = n + 1$ case. Of course the iterates produced by the chord method and their limit (if it exists) must also lie in this affine subspace. In the case at hand, this affine subspace is just a line in $\mathbf{R}^2$ which has a unique point of intersection with the cubic, and the iterates of both methods converge to this point.

It follows from these observations that in general *the iterates produced by Algorithm 2.4 with the first Broyden update (4.1) cannot possibly converge to a solution of Problem 1.1 if the affine subspace* $\bar{x}_0 + \text{range}\,\{B_0^T\}$ *does not intersect the solution set.* It can be seen from (4.2) that Algorithm 2.8 with the second Broyden update does not share this potential disadvantage. To show that this disadvantage can be realized in practice, we took $F(\bar{x}) = x_1^2 - x_2$, the zero curve of which is a parabola through the origin, and considered the starting point $\bar{x}_0 = (1, -1)$. The affine subspace $\bar{x}_0 + \text{range}\,\{B_0^T\}$ for $B_0 = F'(\bar{x}_0)$ is the line $(1, -1) + t(2, -1)$, $-\infty < t < \infty$, which does not intersect the parabola, and so the iterates produced by Algorithm 2.4 with the update (4.1) and with $B_0 = F'(\bar{x}_0)$ cannot possibly converge to a solution from this starting point. The same is true for the chord method. However, Algorithm 1.3 and Algorithm 2.8 with the update (4.2) did yield solutions. The results are summarized in Table 3.

Method	Number of Iterations	Final Iterate
Algorithm 1.3	4	(-.01868,.0003489)
Algorithm 2.4	*	*
Algorithm 2.8	16	(.1985,.03942)
Chord Method	*	*

* unable to converge

Table 3. Results for $F(\bar{x}) = x_1^2 - x_2$ with $\bar{x}_0 = (1, -1)$.

The above results and observations seem to indicate that the second Broyden update (4.2) may have advantages over the first Broyden update (4.1). This may be the case in some circumstances, but the update (4.2) also has possibly unattractive aspects. For one thing, it distinguishes a particular subset of the columns of the matrices being updated, which may be undesirable for a variety of reasons. For another, it may be poorly suited for use in many homotopy algorithms. We have in mind predictor-corrector algorithms in which each predictor step is in a direction approximately tangent to the homotopy zero curve being traced and is followed by a series of corrector steps determined by an algorithm of the type considered here. It is usually desirable to update the current approximate Jacobian following a predictor step as well as after each corrector step. If we denote the current approximate Jacobian at a predictor step by B and the step in an approximate tangent direction by $\bar{s}$, then $B\bar{s} = 0$ and the denominator of (4.2) is just $t^T t$ $(= t^2$ in the $\bar{n} = n + 1$ case). If one is near a point at which t is small, then this denominator may be very small and the update determined by (4.2) may be numerically unstable. This instability may be compounded in the subsequent corrector iterations: Since the iterates are not constrained to a manifold, it may be difficult to control them and guarantee forward progress along the homotopy zero curve. We note that the first Broyden update (4.1) does not have these potential flaws; indeed, Georg [15] has observed that it may be particularly well-suited for updating after a step in an approximate tangent direction in that it effectively incorporates current "tangent information" in the updated approximate Jacobian. However, as we observe above, the update (4.1) cannot incorporate current tangent information after a corrector step. It seems that what is needed is an update which effectively incorporates tangent information on both predictor and corrector steps, does not distinguish a particular subset of the columns of the matrices being updated, and yet yields controllable iterates.

In our second set of experiments, we addressed the effectiveness of the algorithms of interest here in performing the corrector iterations in a highly-developed homotopy method code applied to a real test problem. The code is the HOMPACK suite [31], which allows the use of predictor-corrector methods with either the normal flow algorithm or an augmented Jacobian algorithm in the corrector steps. (It also offers an ordinary differential equation - based algorithm which we do not consider here.) The normal flow algorithm in HOMPACK is just Algorithm 1.3 with an analytic evaluation of F' at each iteration; we refer to the method using it as NF below. The augmented Jacobian algorithm is Algorithm 3.3 with the first Broyden update (4.1) and with V equal to an approximate tangent vector; we refer to the method using it as AJB1 below. In this algorithm, B_0 is obtained by an analytic evaluation of F' at either the last point on the curve or (if a corrector failure has occurred) the current predictor point. For our experiments we also included modifications of the HOMPACK method using the normal flow algorithm which implement Algorithm 2.4 with the first Broyden update (4.1) and Algorithm 2.8 with the second Broyden update (4.2) in the corrector steps. We refer to these modified methods respectively as NFB1 and NFB2 below. In these methods, B_0 is obtained as in AJB1; all other procedures and strategies, such as step-size selection, are as in NF. We did not use a chord method in these experiments; it is noted in [31] that such methods are rarely cost-effective.

The test problem we used is a real geometric modelling problem that arose at General Motors Research Laboratories and is described in Morgan [21]. While this problem is not challenging for homotopy methods, it is very difficult for globalized Newton-like methods such as those found in MINPACK [18], and it is nontrivial. The goal is to find all zeroes of $G : \mathbf{R}^2 \to \mathbf{R}^2$, where for $x = (x_1, x_2) \in \mathbf{R}^2$, $G(x) = (G_1(x), G_2(x))$ is given by

$$G_j(x) = a_{j1}x_1^2 + a_{j2}x_2^2 + a_{j3}x_1x_2 + a_{j4}x_1 + a_{j5}x_2 + a_{j6} = 0, \qquad \text{for } j = 1, 2,$$

where

$$
\begin{array}{llll}
a_{11} = -.00098 & a_{14} = -235 & a_{21} = -.01 & a_{24} = .00987 \\
a_{12} = 978000 & a_{15} = 88900 & a_{22} = -.984 & a_{25} = -.124 \\
a_{13} = -9.8 & a_{16} = -1.0 & a_{23} = -29.7 & a_{26} = -.25
\end{array}
$$

For a problem such as this in which each component of the nonlinear function is a polynomial, HOMPACK provides a special homotopy algorithm which finds all solutions, real and complex. For this system, the solutions (to four significant figures) are

$$
\begin{aligned}
(x_1, x_2) = &(.09089, -.09115), \\
&(2342, -.7883), \\
&(.01615 + 1.685i, .0002680 + .004428i), \\
&(.01615 - 1.685i, .0002680 - .004428i).
\end{aligned}
$$

We used HOMPACK to solve this test problem, allowing it to construct and track the (four) zero curves of the homotopy map using NF, AJB1, NFB1, and NFB2. In the trials involving NFB1 and NFB2, we allowed up to six corrector iterations before declaring convergence failure, instead of the usual maximum of four in NF, and we used an "ideal" residual reduction factor for the corrector iterations (see [31]) of .5 instead of the usual default factor of .01 in NF. This extra leeway seemed more appropriate and resulted in better performance for the updating methods. The pertinent performance data for finding the four solutions are given in Table 4. We emphasize that these data actually reflect a series of corrector iterations from different starting points. Following corrector convergence failure, the starting points are moved toward the zero curve until convergence occurs, and the cost of corrector failures is not simply ignored but is counted. Such testing is more meaningful for homotopy algorithm evaluation than *in vacuo* tests involving a single set of corrector iterations, since how a scheme performs in conjunction with prediction and step size correction strategies is ultimately more important than its performance in isolation.

Method	Function Evaluations	Jacobian Evaluations	Homotopy Steps
NF	171	171	66
AJB1	483	71	52
NFB1	462	86	82
NFB2	747	190	186

Table 4. HOMPACK results for the geometric modelling problem.

These results suggest that AJB1 and NFB1 are roughly comparable in performance and may be preferable to NF in many circumstances. The performance of NFB2 is clearly inferior to that of the other methods, which suggests that the unattractive features of the second Broyden update (4.2) outweighed its potential advantages in this experiment. A detailed examination of the performance of NFB2 showed that the number of corrector iterations per homotopy step varied between two and over twenty. This inconsistent behavior suggests that this update may be too unreliable for general use in homotopy algorithms, and it would at least make it difficult to use the iteration behavior to estimate optimal step-sizes.

To further observe the behavior of the methods, we allowed HOMPACK to take a single step a distance .5 along the tangent from the initial point of each of the four zero curves of the homotopy map using NF, AJB1, NFB1, and NFB2. Each updating method was allowed just one initial Jacobian evaluation; NF used one Jacobian evaluation per iteration as usual. The methods were allowed to iterate to termination, rather than being restarted closer to the zero curve if convergence did not occur after some maximum number of iterations; thus the number of function evaluations is equal to the number of iterations, which is also the number of Jacobian evaluations for NF. The resulting numbers of function evaluations are given in Table 5. We believe the data for NFB2 in the successful cases are not as encouraging as they appear, since the return to the zero curve for NFB2 may be erratic and may be to an earlier (already traversed) point on the homotopy curve, which is worthless.

Method	Curve 1	Curve 2	Curve 3	Curve 4
NF	4	5	5	4
AJB1	8	37	35	8
NFB1	9	27	38	8
NFB2	3	*	3	3

* produced overflow (divergence)

Table 5. Function evaluations for one HOMPACK step on the geometric modelling problem.

These limited experiments permit some conclusions and clearly indicate where more research is needed. (1) Least-change secant update methods based on the normal flow and augmented Jacobian algorithms are theoretically sound and should be part of the arsenal for solving underdetermined systems. (2) Some of these updates are reasonably efficient, reliable, and numerically stable. (3) More research is needed to understand the tradeoffs between efficiency, reliability, and stability. (4) Extensive numerical testing of the updates described here and in [6] and [3] remains to be done. (5) Updates should be sought for predictor-corrector homotopy algorithms which produce iteration sequences having desirable properties, e. g., by effectively incorporating tangent information on both the predictor and corrector steps without distinguishing columns of the matrices being updated.

Appendix. We first outline a local convergence analysis for a very general algorithm formulated as

ALGORITHM A.1 *Given* $\bar{x}_0 \in \Omega$ *and* $B_0 \in \mathbf{A}$, *determine for* $k = 0, 1, \ldots,$

$$\bar{x}_{k+1} = \bar{x}_k - B_k^+ F(\bar{x}_k),$$
$$B_{k+1} \in U(\bar{x}_k, \bar{x}_{k+1}, B_k).$$

In Algorithm A.1, U is an *update function*, the values of which are subsets of $\mathbf{A}$ and which we assume to be defined on $\Omega \times \Omega \times \mathbf{A}$. In the case of an equal number of equations and unknowns, it is traditional to consider an update function which exhibits a property known as *bounded deterioration* from some fixed distinguished matrix, which is typically some approximation of the Jacobian at a solution of interest. Bounded deterioration amounts to assuming that while Jacobian approximations may not get better as the iterations proceed, they at least only get worse sufficiently slowly to allow local q-linear convergence. In the analysis below, we assume that the update function in Algorithm A.1 exhibits bounded deterioration from F' at the current point in that the following hypothesis is satisfied. Note that in this hypothesis, the "deterioration" is determined by the length of the step $\bar{s} = \bar{x}_+ - \bar{x}$, rather than by the traditional distances from $\bar{x}$ and $\bar{x}_+$ to a solution as in [14].

HYPOTHESIS A.2. *There are nonnegative constants* α_1 *and* α_2 *such that for each* $(\bar{x}, \bar{x}_+, B) \in \Omega \times \Omega \times \mathbf{A}$, *every* $B_+ \in U(\bar{x}, \bar{x}_+, B)$ *satisfies*

$$\|B_+ - F'(\bar{x}_+)\| \leq (1 + \alpha_1 |\bar{s}|^p) \|B - F'(\bar{x})\| + \alpha_2 |\bar{s}|^p,$$

where $\bar{s} = \bar{x}_+ - \bar{x}$.

Proposition A.3 below is an elementary technical result which we have used in previous sections and which we use in the convergence analysis that follows.

PROPOSITION A.3. *Under Hypothesis 1.2, one has*

$$|F(\bar{y}) - F(\bar{x}) - B(\bar{y} - \bar{x})| \leq \left\{ \frac{\gamma}{1+p} |\bar{y} - \bar{x}|^p + |B - F'(\bar{x})| \right\} |\bar{y} - \bar{x}|$$

for any $\bar{x}$, $\bar{y} \in \Omega$ *and* $B \in \mathbf{R}^{n \times \bar{n}}$.

Proof. Setting $\bar{x}(t) = \bar{x} + t(\bar{y} - \bar{x})$, one has

$$\begin{aligned} |F(\bar{y}) - F(\bar{x}) - B(\bar{y} - \bar{x})| &= \left| \left\{ \int_0^1 [F'(\bar{x}(t)) - F'(\bar{x})]\, dt + [F'(\bar{x}) - B] \right\} (\bar{y} - \bar{x}) \right| \\ &\leq \left\{ \frac{\gamma}{1+p} |\bar{y} - \bar{x}|^p + |B - F'(\bar{x})| \right\} |\bar{y} - \bar{x}|. \qquad \square \end{aligned}$$

In the following, we assume that ξ_1 and ξ_2 are positive constants such that $\xi_1 |M| \leq \|M\| \leq \xi_2 |M|$ for all $M \in \mathbf{R}^{n \times \bar{n}}$.

THEOREM A.4. *Let F satisfy Hypothesis 1.2, suppose Ω_η is given by (1.1) for some $\eta > 0$, and let U satisfy Hypothesis A.2. For any $r \in (0,1)$ and $\mu' > \mu$, there exist $\epsilon > 0$ and $\delta > 0$ such that if $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ satisfy $|F(\bar{x}_0)| < \epsilon$ and $|B_0 - F'(\bar{x}_0)| < \delta$, then the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ determined by Algorithm A.1 are well-defined and converge q-linearly to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$ with*

$$|\bar{x}_{k+1} - \bar{x}_*| \leq r|\bar{x}_k - \bar{x}_*|, \qquad k = 0, 1, \ldots, \tag{A.1}$$

and with $\{|B_k^+|\}_{k=0,1,\ldots}$ uniformly bounded by μ'. Also, $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ is uniformly small.

Proof. Suppose $r \in (0,1)$ and $\mu' > \mu$ are given. Let $\delta' > 0$ be such that if $|B - F'(\bar{x})| < \delta'$ for $B \in \mathbf{A}$ and $\bar{x} \in \Omega_{\eta/2}$, then B is of full rank n and $|B^+| \leq \mu'$. Then for $\bar{x} \in \Omega_{\eta/2}$ and $B \in \mathbf{A}$ such that $|B - F'(\bar{x})| < \delta'$, $\bar{s} = -B^+F(\bar{x})$ is well-defined and

$$|\bar{s}| \leq \mu'|F(\bar{x})|. \tag{A.2}$$

If also $\bar{x}_+ = \bar{x} + \bar{s} \in \Omega_{\eta/2}$ and $|B_+ - F'(\bar{x}_+)| < \delta'$ for $B_+ \in \mathbf{A}$, then $\bar{s}_+ = -B_+^+F(\bar{x}_+)$ is well-defined, and (A.2) and Proposition A.3 (with $\bar{y} = \bar{x}_+$) give

$$\begin{aligned} |\bar{s}_+| &\leq \mu'|F(\bar{x}_+)| \\ &\leq \mu'|F(\bar{x}_+) - F(\bar{x}) - B\bar{s}| \\ &\leq \mu' \left\{ \frac{\gamma}{1+p}|\bar{s}|^p + |B - F'(\bar{x})| \right\} |\bar{s}|. \end{aligned} \tag{A.3}$$

Suppose M is such that $|F'(\bar{x})| \leq M$ for $\bar{x} \in \Omega_{\eta/2}$. We further restrict δ' if necessary so that $\mu'\delta' < 1$ and

$$\frac{\mu'^2\delta' M}{1 - \mu'\delta'} < r.$$

We now take $\delta > 0$, $\epsilon > 0$ so small that

$$\rho \equiv \mu' \left\{ \frac{\gamma(\mu'\epsilon)^p}{1+p} + \delta' \right\} < 1, \tag{A.4.1}$$

$$\frac{\mu'\rho}{1-\rho} \left\{ M + \frac{\gamma}{1+p} \left(\frac{\mu'\epsilon}{1-\rho} \right)^p \right\} \leq r, \tag{A.4.2}$$

$$\frac{\mu'\epsilon}{1-\rho} < \frac{\eta}{2}, \tag{A.4.3}$$

$$\frac{\xi_2}{\xi_1}\delta + \frac{(\alpha_1\xi_2\delta' + \alpha_2)(\mu'\epsilon)^p}{\xi_1(1-\rho^p)} < \delta'. \tag{A.4.4}$$

Note that since $\xi_1 \leq \xi_2$, (A.4.4) implies $\delta < \delta'$

Suppose $\bar{x}_0 \in \Omega_\eta$ and $B_0 \in \mathbf{A}$ are such that $|F(\bar{x}_0)| < \epsilon$ and $|B_0 - F'(\bar{x}_0)| < \delta$. Then $\bar{s}_0 = -B_0^+ F(\bar{x}_0)$ is well-defined and $(A.2)$ gives $|\bar{s}_0| \le \mu'\epsilon$, which we use often below. From $(A.4.3)$, one has $\bar{x}_1 = \bar{x}_0 + \bar{s}_0 \in \Omega_{\eta/2}$, and Hypothesis A.2 gives

$$\|B_1 - F'(\bar{x}_1)\| \le \|B_0 - F'(\bar{x}_0)\| + (\alpha_1\xi_2\delta + \alpha_2)|\bar{s}_0|^p,$$

which with $(A.4.4)$ implies

$$|B_1 - F'(\bar{x}_1)| \le \frac{\xi_2}{\xi_1}\delta + \frac{(\alpha_1\xi_2\delta' + \alpha_2)(\mu'\epsilon)^p}{\xi_1(1-\rho^p)} < \delta'.$$

Then $\bar{s}_1 = -B_1^+ F(\bar{x}_1)$ and $\bar{x}_2 = \bar{x}_1 + \bar{s}_1$ are well-defined, and it follows from $(A.3)$ and $(A.4.1)$ that

$$|\bar{s}_1| \le \mu' \left\{ \frac{\gamma}{1+p}|\bar{s}_0|^p + |B_0 - F'(\bar{x}_0)| \right\} |\bar{s}_0| \le \rho|\bar{s}_0|.$$

As an inductive hypothesis, assume that for some $k > 0$ and for $j = 0, \ldots, k$, one has B_j satisfying $|B_j - F'(\bar{x}_j)| < \delta'$ and $\bar{s}_j = \bar{x}_{j+1} - \bar{x}_j = -B_j^+ F(\bar{x}_j)$ satisfying $|\bar{s}_j| \le \rho|\bar{s}_{j-1}|$ if $j > 0$. Then $(A.4.3)$ gives

$$|\bar{x}_{k+1} - \bar{x}_0| \le \sum_{j=0}^{k} |\bar{s}_j| \le \frac{\mu'\epsilon}{1-\rho} < \frac{\eta}{2},$$

and so $\bar{x}_{k+1} \in \Omega_{\eta/2}$. Also, Hypothesis A.2 gives for $j = 0, \ldots, k$,

$$\begin{aligned}\|B_{j+1} - F'(\bar{x}_{j+1})\| &\le \|B_j - F'(\bar{x}_j)\| + (\alpha_1\xi_2\delta' + \alpha_2)|\bar{s}_j|^p \\ &\le \|B_j - F'(\bar{x}_j)\| + (\alpha_1\xi_2\delta' + \alpha_2)(\mu'\epsilon)^p\rho^{pj},\end{aligned}$$

which yields

$$\|B_{k+1} - F'(\bar{x}_{k+1})\| \le \|B_0 - F'(\bar{x}_0)\| + (\alpha_1\xi_2\delta' + \alpha_2)(\mu'\epsilon)^p \sum_{j=0}^{k} \rho^{pj}$$

and, with $(A.4.4)$,

$$|B_{k+1} - F'(\bar{x}_{k+1})| \le \frac{\xi_2}{\xi_1}\delta + \frac{(\alpha_1\xi_2\delta' + \alpha_2)(\mu'\epsilon)^p}{\xi_1(1-\rho^p)} < \delta'.$$

Then $\bar{s}_{k+1} = -{B_{k+1}}^+ F(\bar{x}_{k+1})$ and $\bar{x}_{k+2} = \bar{x}_{k+1} + \bar{s}_{k+1}$ are well-defined, and $(A.3)$ and $(A.4.1)$ give

$$(A.5) \qquad |\bar{s}_{k+1}| \le \mu' \left\{ \frac{\gamma}{1+p}|\bar{s}_k|^p + |B_k - F'(\bar{x}_k)| \right\} |\bar{s}_k| \le \rho|\bar{s}_k|.$$

With this induction, one sees that the iterates $\{\bar{x}_k\}_{k=0,1,\ldots}$ are well-defined, remain in $\Omega_{\eta/2}$, and constitute a Cauchy sequence with limit $\bar{x}_* \in \Omega$. Since $(A.3)$ implies

$$|F(\bar{x}_{k+1})| \leq \left\{ \frac{\gamma(\mu'\epsilon)^p}{1+p} + \delta' \right\} |\bar{s}_k|, \tag{A.6}$$

one has $F(\bar{x}_*) = 0$. Furthermore, $|B_k - F'(\bar{x}_k)| < \delta'$ for $k = 0, 1, \ldots,$ and so $\{|B_k^+|\}_{k=0,1,\ldots}$ is uniformly bounded by μ' and $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ is uniformly small.

To show that $(A.1)$ holds, we note that Proposition A.3 (with $\bar{x} = \bar{x}_k$, $\bar{y} = \bar{x}_{k+1}$, and $B = B_k$) and $(A.2)$ give

$$\begin{aligned} |F(\bar{x}_{k+1})| &= |F(\bar{x}_{k+1}) - F(\bar{x}_k) - B_k \bar{s}_k| \\ &\leq \left\{ \frac{\gamma}{1+p} |\bar{s}_k|^p + |B_k - F'(\bar{x}_k)| \right\} |\bar{s}_k| \\ &\leq \rho |F(\bar{x}_k)|. \end{aligned} \tag{A.7}$$

Then Proposition A.3 (with $\bar{x} = \bar{x}_*$, $\bar{y} = \bar{x}_k$, and $B = F'(\bar{x}_*)$) gives

$$\begin{aligned} |F(\bar{x}_k)| &\leq |F'(\bar{x}_*)||\bar{x}_k - \bar{x}_*| + \frac{\gamma}{1+p} |\bar{x}_k - \bar{x}_*|^{1+p} \\ &\leq \left\{ M + \frac{\gamma}{1+p} |\bar{x}_k - \bar{x}_*|^p \right\} |\bar{x}_k - \bar{x}_*|. \end{aligned} \tag{A.8}$$

Also, $(A.2)$ gives

$$\mu' |F(\bar{x}_{k+1})| \geq |\bar{s}_{k+1}|. \tag{A.9}$$

One has from $(A.5)$ that

$$\begin{aligned} |\bar{s}_{k+1}| &\geq |\bar{x}_{k+1} - \bar{x}_*| - |\bar{x}_{k+2} - \bar{x}_*| \\ &\geq |\bar{x}_{k+1} - \bar{x}_*| - \sum_{j=k+2}^{\infty} |\bar{s}_j| \\ &\geq |\bar{x}_{k+1} - \bar{x}_*| - \frac{\rho}{1-\rho} |\bar{s}_{k+1}|, \end{aligned}$$

and so

$$|\bar{s}_{k+1}| \geq (1-\rho) |\bar{x}_{k+1} - \bar{x}_*|. \tag{A.10}$$

It follows from $(A.7)$, $(A.8)$, $(A.9)$, and $(A.10)$ that

$$|\bar{x}_{k+1} - \bar{x}_*| \leq \frac{\mu'\rho}{1-\rho} \left\{ M + \frac{\gamma}{1+p} |\bar{x}_k - \bar{x}_*|^p \right\} |\bar{x}_k - \bar{x}_*|, \tag{A.11}$$

and $(A.1)$ follows from $(A.11)$, $(A.5)$, and $(A.4.2)$. $\square$

Remark. It is apparent from the proof that ϵ and δ depend only on r, μ', γ, p, η, α_1 and α_2 of Hypothesis A.2, and the bound M on $|F'(\bar{x})|$ for $\bar{x} \in \Omega_{\eta/2}$ (as well as on ξ_1 and ξ_2 which depend on $|\cdot|$ and $\|\cdot\|$).

THEOREM A.5. *Suppose that F satisfies Hypothesis 1.2 and that $\{\bar{x}_k\}_{k=0,1,\ldots}$ is a sequence generated by Algorithm A.1 which converges q-linearly to $\bar{x}_* \in \Omega$ with $(A.1)$ satisfied for some $r \in (0,1)$ and with $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k \neq 0$ for all k. Then for any $B_* \in \mathbf{R}^{n\times\bar{n}}$,*

$$\lim_{k\to\infty} \frac{|(B_k - B_*)\bar{s}_k|}{|\bar{s}_k|} = 0 \tag{A.12}$$

if and only if

$$\lim_{k\to\infty} \frac{|[B_* - F'(\bar{x}_*)](\bar{x}_k - \bar{x}_*) - B_*(\bar{x}_{k+1} - \bar{x}_*)|}{|\bar{x}_k - \bar{x}_*|} = 0. \tag{A.13}$$

Proof. One has

$$(B_k - B_*)\bar{s}_k = [B_* - F'(\bar{x}_*)](\bar{x}_k - \bar{x}_*) - B_*(\bar{x}_{k+1} - \bar{x}_*) + F'(\bar{x}_*)(\bar{x}_k - \bar{x}_*) - F(\bar{x}_k). \tag{A.14}$$

It follows from Proposition A.3 (with $\bar{x} = \bar{x}_*$, $\bar{y} = \bar{x}_k$, and $B = F'(\bar{x}_*)$) that

$$F'(\bar{x}_*)(\bar{x}_k - \bar{x}_*) - F(\bar{x}_k) = o(|\bar{x}_k - \bar{x}_*|). \tag{A.15}$$

Furthermore,

$$(1-r)|\bar{x}_k - \bar{x}_*| \le |\bar{s}_k| \le (1+r)|\bar{x}_k - \bar{x}_*|. \tag{A.16}$$

It follows immediately from $(A.14)$, $(A.15)$, and $(A.16)$ that $(A.12)$ holds if and only if $(A.13)$ holds. $\square$

Note that $(A.12)$ and $(A.13)$ are *norm-independent* in that if either holds in any pair of norms on $\mathbf{R}^n$ and $\mathbf{R}^{\bar{n}}$, then it also holds in every pair of norms on $\mathbf{R}^n$ and $\mathbf{R}^{\bar{n}}$. In Proposition A.6 below, we summarize some particular consequences of Theorem A.5. In Proposition A.6, both $(A.17)$ and the property of q-superlinear convergence are norm-independent; however, the size of r on which q-superlinear convergence is conditioned is norm-dependent.

PROPOSITION A.6. *Suppose that the hypotheses of Theorem A.5 are satisfied and that $(A.12)$ holds. Then*

$$\overline{\lim_{k\to\infty}} \frac{|B_*{}^+ B_*(\bar{x}_{k+1} - \bar{x}_*)|}{|\bar{x}_k - \bar{x}_*|} = \overline{\lim_{k\to\infty}} \frac{|B_*{}^+[B_* - F'(\bar{x}_*)](\bar{x}_k - \bar{x}_*)|}{|\bar{x}_k - \bar{x}_*|} \le |B_*{}^+[B_* - F'(\bar{x}_*)]|. \tag{A.17}$$

It follows that if $\{|B_k^+|\}_{k=0,1,\ldots}$ *is uniformly bounded by* μ' *and if* $r < (1+\mu'|B_*|)^{-1}$, *then*

$$(A.18) \qquad \lim_{k\to\infty} \frac{|\bar{x}_{k+1}-\bar{x}_*|}{|\bar{x}_k-\bar{x}_*|} \le \frac{\mu'|B_*|}{1-r(1+\mu'|B_*|)} \left|B_*{}^+\left[B_*-F'(\bar{x}_*)\right]\right|.$$

In particular, if $B_* = F'(\bar{x}_*)$ *as well, then* $\{\bar{x}_k\}_{k=0,1,\ldots}$ *converges q-superlinearly to* $\bar{x}_*$.

Proof. If $(A.12)$ holds, then $(A.13)$ also holds and implies

$$\lim_{k\to\infty} \frac{\left|B_*{}^+\left[B_*-F'(\bar{x}_*)\right](\bar{x}_k-\bar{x}_*) - B_*{}^+B_*(\bar{x}_{k+1}-\bar{x}_*)\right|}{|\bar{x}_k-\bar{x}_*|} = 0,$$

from which $(A.17)$ follows immediately. Suppose $\{|B_k^+|\}_{k=0,1,\ldots}$ is uniformly bounded by μ'. Then

$$(A.19) \qquad \begin{aligned} \frac{\left|B_*{}^+B_*(\bar{x}_{k+1}-\bar{x}_*)\right|}{|\bar{x}_k-\bar{x}_*|} &= \frac{\left|B_*{}^+\left\{B_{k+1}\bar{s}_{k+1} + (B_*-B_{k+1})\bar{s}_{k+1} - B_*(\bar{x}_{k+2}-\bar{x}_*)\right\}\right|}{|\bar{x}_k-\bar{x}_*|} \\ &\ge \frac{|B_{k+1}\bar{s}_{k+1}|}{|B_*||\bar{x}_k-\bar{x}_*|} - \left|B_*{}^+\right| \frac{|(B_*-B_{k+1})\bar{s}_{k+1}|}{|\bar{x}_k-\bar{x}_*|} \\ &\quad - \frac{|\bar{x}_{k+2}-\bar{x}_*|}{|\bar{x}_k-\bar{x}_*|} \\ &\ge \frac{(1-r)}{\mu'|B_*|}\frac{|\bar{x}_{k+1}-\bar{x}_*|}{|\bar{x}_k-\bar{x}_*|} - r(1+r)\left|B_*{}^+\right| \frac{|(B_*-B_{k+1})\bar{s}_{k+1}|}{|\bar{s}_{k+1}|} \\ &\quad - r\frac{|\bar{x}_{k+1}-\bar{x}_*|}{|\bar{x}_k-\bar{x}_*|}. \end{aligned}$$

If $r < (1+\mu'|B_*|)^{-1}$, then $(A.19)$, $(A.12)$, and $(A.17)$ imply $(A.18)$. □

We now outline a local convergence analysis for an inverse-update analogue of Algorithm A.1, which we formulate as

ALGORITHM A.7. *Given* $\bar{x}_0 \in \Omega$ *and* $B_0 = [\mathcal{B}_0, \mathcal{C}_0]$ *with* $[\mathcal{B}_0^{-1}, -\mathcal{B}_0^{-1}\mathcal{C}_0] \in \mathbf{A}$, *determine for* $k = 0, 1, \ldots,$

$$\begin{aligned} \bar{x}_{k+1} &= \bar{x}_k - B_k^+F(\bar{x}_k), \\ K_k &= \left[\mathcal{B}_k^{-1}, -\mathcal{B}_k^{-1}\mathcal{C}_k\right], \ \textit{where } B_k = [\mathcal{B}_k, \mathcal{C}_k], \\ K_{k+1} &\in U(\bar{x}_k, \bar{x}_{k+1}, K_k), \\ B_{k+1} &= \left[\mathcal{K}_{k+1}^{-1}, -\mathcal{K}_{k+1}^{-1}\mathcal{L}_{k+1}\right], \ \textit{where } K_{k+1} = [\mathcal{K}_{k+1}, \mathcal{L}_{k+1}]. \end{aligned}$$

As in Algorithm A.1, U is an update function, the values of which are subsets of $\mathbf{A}$ and which is defined on $\Omega \times \Omega \times \mathbf{A}$. The bounded deterioration assumption which is now appropriate for U is formulated in the following hypothesis.

HYPOTHESIS A.8. *$F_x(\bar{x})$ is nonsingular for all $\bar{x} \in \Omega$, and there are nonnegative constants α_1 and α_2 such that for each $(\bar{x}, \bar{x}_+, K) \in \Omega \times \Omega \times \mathbf{A}$, every $K_+ \in U(\bar{x}, \bar{x}_+, K)$ satisfies*

$$\|K_+ - [F_x(\bar{x}_+)^{-1}, -F_x(\bar{x}_+)^{-1}F_\lambda(\bar{x}_+)]\| \leq \\ (1+\alpha_1|\bar{s}|^p)\,\|K - [F_x(\bar{x})^{-1}, -F_x(\bar{x})^{-1}F_\lambda(\bar{x})]\| + \alpha_2|\bar{s}|^p,$$

where $\bar{s} = \bar{x}_+ - \bar{x}$.

Theorems A.9 and A.10 below are analogues for Algorithm A.7 of Theorems A.4 and A.5 for Algorithm A.1. The proof of Theorem A.9 is similar to that of Theorem A.4, and we omit it.

THEOREM A.9. *Let F and U satisfy Hypotheses 1.2 and A.8, and suppose Ω_η is given by (1.1) for some $\eta > 0$. For any $r \in (0,1)$ and $\mu' > \mu$, there exist $\epsilon > 0$ and $\delta > 0$ such that if $\bar{x}_0 \in \Omega_\eta$ and $B_0 = [\mathcal{B}_0, \mathcal{C}_0]$ with $[\mathcal{B}_0^{-1}, -\mathcal{B}_0^{-1}\mathcal{C}_0] \in \mathbf{A}$ satisfy $|F(\bar{x}_0)| < \epsilon$ and $|B_0 - F'(\bar{x}_0)| < \delta$, then the iterates $\{\bar{x}_k\}_{k=0,1,\dots}$ determined by Algorithm A.7 are well-defined and converge q-linearly to a point $\bar{x}_* \in \Omega$ such that $F(\bar{x}_*) = 0$ with*

$$(A.20) \qquad |\bar{x}_{k+1} - \bar{x}_*| \leq r|\bar{x}_k - \bar{x}_*|, \qquad k = 0, 1, \dots,$$

and with $\{|B_k^+|\}_{k=0,1,\dots}$ uniformly bounded by μ'. Also, $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\dots}$ is uniformly small.

THEOREM A.10. *Suppose that F satisfies Hypothesis 1.2, that $F_x(\bar{x})$ is nonsingular for all $\bar{x} \in \Omega$, and that $\{\bar{x}_k = (x_k, \lambda_k)\}_{k=0,1,\dots}$ is a sequence generated by Algorithm A.7 which converges q-linearly to $\bar{x}_* \in \Omega$ with (A.20) satisfied for some $r \in (0,1)$, with $\bar{s}_k = \bar{x}_{k+1} - \bar{x}_k \neq 0$ for all k, and with $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\dots}$ uniformly small. Let $K_* = [\mathcal{K}_*, \mathcal{L}_*] \in \mathbf{R}^{n\times\bar{n}}$ be any matrix such that $\mathcal{K}_*$ is invertible and suppose that $\{y_k\}_{k=0,1,\dots}$ satisfies*

$$(A.21) \qquad |K_*\bar{y}_k - s_k| \leq \alpha_k|\bar{s}_k|, \qquad k = 0, 1, \dots,$$

where $\bar{y}_k = (y_k, \lambda_{k+1} - \lambda_k)$, $s_k = x_{k+1} - x_k$, and $\lim_{k\to\infty} \alpha_k = 0$. Then

$$(A.22) \qquad \lim_{k\to\infty} \frac{|(K_k - K_*)\bar{y}_k|}{|\bar{y}_k|} = 0$$

if and only if (A.13) holds with $B_* = [\mathcal{K}_*^{-1}, -\mathcal{K}_*^{-1}\mathcal{L}_*]$.

Remark. If the hypotheses of Theorem A.10 are satisfied and (A.22) holds, then one easily verifies that all conclusions of Proposition A.6 hold with $B_* = [\mathcal{K}_*^{-1}, -\mathcal{K}_*^{-1}\mathcal{L}_*]$.

Proof. We show that under the hypotheses of the theorem, (A.22) is equivalent to (A.12) with $B_* = [\mathcal{K}_*^{-1}, -\mathcal{K}_*^{-1}\mathcal{L}_*]$, from which the equivalence of (A.13) follows. One has

$$(A.23) \qquad (K_k - K_*)\bar{y}_k = -\mathcal{K}_k(B_k - B_*)\bar{s}_k + \left(I - \mathcal{K}_k\mathcal{K}_*^{-1}\right)(s_k - K_*\bar{y}_k),$$

and $(A.21)$ implies there are positive constants η_1 and η_2 for which

$$\eta_1|\bar{s}_k| \leq |\bar{y}_k| \leq \eta_2|\bar{s}_k|, \qquad k = 0, 1, \ldots. \tag{A.24}$$

If $\{|B_k - F'(\bar{x}_k)|\}_{k=0,1,\ldots}$ is uniformly sufficiently small that $\{|\mathcal{B}_k|\}_{k=0,1,\ldots}$ and $\{|\mathcal{K}_k|\}_{k=0,1,\ldots}$ are uniformly bounded, then $(A.21)$, $(A.23)$, and $(A.24)$ imply the equivalence of $(A.22)$ and $(A.12)$ with $B_* = [\mathcal{K}_*^{-1}, -\mathcal{K}_*^{-1}\mathcal{L}_*]$. □

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