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Introduction

Large space structures are much more flexible than their terrestrial counterparts[1]. Rotation of large space structures may be desirable for stability, thermal or artificial gravity reasons. Previous literature includes the large deformations due to the free rotation of a slender rod[2,3] and a ring about a diameter[4].

An important model of a space station is a ring rotating about its axis of symmetry. If the ring is balanced it will be stable and remain circular. The present note considers the case when the ring is unbalanced by a mass attached to a point on the ring.

Formulation

Figure 1 shows the origin of the coordinate system located on the symmetry line opposite a mass of magnitude m. The system is rotating in its own plane with angular velocity Ω about the center of mass. Normalize all lengths by the natural radius R of the ring and all stresses by EI/R^3 where EI is the flexural rigidity. The governing equation for large deformations is[5]

$$\theta'\theta'''' - \theta''\theta''' + \theta''(\theta')^3 - q_n\theta'' + q_t(\theta')^2 + q_n'\theta' = 0. \tag{1}$$

Here $\theta(s)$ is the local inclination, s is the arc length from the origin, and q_n and q_t are normal and tangential stresses caused by centrifugal forces. Consider an elemental length shown in Figure 1. Let ρ be the mass per unit length of the ring. We find

$$q_t = -Br\,\sin(\theta - \psi), \qquad q_n = Br\,\cos(\theta - \psi) \tag{2}$$

where

$$r\,\cos\psi = h - y, \qquad r\,\sin\psi = x \tag{3}$$

and $B = \rho\Omega^2 R^4/EI$ is the important nondimensional parameter representing the relative importance of rotation to flexibility. After some work Eq. (1) becomes

$$\begin{aligned}\theta'\theta'''' - \theta''\theta''' + \theta''(\theta')^3 = B\Big\{&\theta''\big[(h-y)\cos\theta + x\,\sin\theta\big] \\ &+ 2(\theta')^2\big[(h-y)\sin\theta - x\,\cos\theta\big]\Big\}.\end{aligned} \tag{4}$$

Notice that h, the distance from the origin to the center of mass, is yet to be determined. The coordinates are related by

$$x' = \cos\theta, \qquad y' = \sin\theta. \tag{5}$$

The boundary conditions for half the ring are

$$\theta(0) = x(0) = y(0) = \theta''(0) = 0, \tag{6}$$

$$x(\pi) = 0, \quad \theta(\pi) = \pi. \tag{7}$$

Balancing shear force at $s = \pi$ gives

$$\alpha \equiv \frac{m}{2\pi\rho R} = \frac{\theta''(\pi)}{\pi B[y(\pi) - h]}. \tag{8}$$

Here α, the mass ratio, is the other important parameter. Given B, α, the sixth order system Eqs. (4, 5) with h a variable is to be solved with the seven boundary conditions Eqs. (6–8).

Perturbation for Small B

If B is small we can perturb from the circular state as follows:

$$\theta = s + B\,\phi(s) + \mathcal{O}(B^2), \tag{9}$$

$$x = \sin s + B\,\xi(s) + \mathcal{O}(B^2), \quad y = 1 - \cos s + B\,\eta(s) + \mathcal{O}(B^2). \tag{10}$$

The leading orders of Eqs. (4–7) are

$$\phi'''' + \phi'' = 2(h-1)\sin s, \tag{11}$$

$$\xi' = -\phi \sin s, \qquad \eta' = \phi \cos s, \tag{12}$$

$$\phi(0) = \xi(0) = \eta(0) = \phi''(0) = \xi(\pi) = \phi(\pi) = 0, \tag{13}$$

$$\alpha = \frac{\phi''(\pi)}{\pi(2-h)}. \tag{14}$$

The solution is

$$\phi = \frac{\alpha}{\alpha+1}\left(s\cos s + s - \frac{3}{2}\sin s\right), \tag{15}$$

$$\xi = \frac{\alpha}{\alpha+1}\left(\frac{1}{4}s\cos 2s - \frac{1}{8}\sin 2s + s\cos s - \sin s + \frac{3}{4}s - \frac{3}{8}\sin 2s\right), \tag{16}$$

$$\eta = \frac{\alpha}{\alpha+1}\left(\frac{1}{4}s\sin 2s + \frac{1}{2}\cos 2s + s\sin s + \cos s + \frac{s^2}{4} - \frac{3}{2}\right), \tag{17}$$

$$h = \frac{2\alpha+1}{\alpha+1}. \tag{18}$$

Thus the normalized moments are

$$\theta'(0) = 1 + \frac{\alpha}{2(\alpha+1)}B + \mathcal{O}(B^2), \tag{19}$$

$$\theta'(\pi) = 1 + \frac{3\alpha}{2(\alpha+1)}B + \mathcal{O}(B^2). \tag{20}$$

The maximum length b is

$$b = y(\pi) = 2 + \frac{\alpha}{\alpha+1}\left(\frac{\pi^2}{4} - 2\right)B + \mathcal{O}(B^2). \tag{21}$$

For the maximum width a, first find the location $s = s^*$ such that $\theta(s^*) = \pi/2$. From Eqs. (9, 15),

$$s^* = \frac{\pi}{2} - \frac{\alpha}{2(\alpha+1)}(\pi - 3)B + \mathcal{O}(B^2). \tag{22}$$

Then

$$a = 2x(s^*) = 2 - \frac{2\alpha}{\alpha+1}\left(1 - \frac{\pi}{4}\right)B + O(B^2). \tag{23}$$

Numerical Integration

For general B, Eqs. (4–7) are converted to a nonlinear system of equations as follows: let

$$v = \begin{pmatrix} \theta'(0) \\ \theta'''(0) \end{pmatrix} \tag{24}$$

and let $x(\eta; v)$, $y(\eta; v)$, $\theta(\eta; v)$ denote the solution to the initial value problem with initial conditions Eqs. (6, 24), assuming h is fixed. Then the two-point boundary value problem (for fixed h) is equivalent to the problem of finding the initial conditions v such that

$$F(v) = \begin{pmatrix} x(\pi; v) \\ \theta(\pi; v) - \pi \end{pmatrix} = 0. \tag{25}$$

Note that $F(v)$ is an implicit highly nonlinear function of v. The system of equations is solved by a globally convergent homotopy method, which has been described theoretically[6] and algorithmically[7]. The mathematical software package HOMPACK[7] was used to perform the computations. This homotopy approach in conjunction with nonstandard shooting has been described elsewhere[8], and so will not be repeated here.

Solving the nonlinear system requires partial derivatives such as $\partial x(\pi; v)/\partial v_1$ (or approximations thereof). These partial derivatives can be accurately and efficiently computed as follows: let

$$Y = \left(x, y, \theta, \theta', \theta'', \theta''', \frac{\partial x}{\partial v_i}, \frac{\partial y}{\partial v_i}, \frac{\partial \theta}{\partial v_i}, \frac{\partial \theta'}{\partial v_i}, \frac{\partial \theta''}{\partial v_i}, \frac{\partial \theta'''}{\partial v_i}\right), \tag{26}$$

which is the solution of the first order initial value problem

$$\begin{aligned}
Y_1' &= \cos Y_3, & Y_7' &= (-\sin Y_3)Y_9, \\
Y_2' &= \sin Y_3, & Y_8' &= (\cos Y_3)Y_9, \\
Y_3' &= Y_4, & Y_9' &= Y_{10}, \\
Y_4' &= Y_5, & Y_{10}' &= Y_{11}, \\
Y_5' &= Y_6, & Y_{11}' &= Y_{12}, \\
Y_6' &= T/Y_4, & Y_{12}' &= \left(Y_4 T' - T Y_{10}\right)/Y_4^2,
\end{aligned} \tag{27}$$

$$Y(0) = \left(0, 0, 0, v_1, 0, v_2, 0, 0, 0, \delta_{1i}, 0, \delta_{2i}\right),$$

where

$$\begin{aligned} T = Y_5 Y_6 - Y_5 Y_4^3 + B\Big\{ & Y_5\left[(h - Y_2)\cos Y_3 + Y_1 \sin Y_3\right] \\ & + 2Y_4^2\left[(h - Y_2)\sin Y_3 - Y_1 \cos Y_3\right]\Big\} \end{aligned}$$

and

$$
\begin{aligned}
T' = {} & Y_5(Y_{12} - 3Y_4^2 Y_{10}) + Y_{11}(Y_6 - Y_4^3) \\
& + B\Big\{ Y_5\big[(Y_2 - h)(\sin Y_3)Y_9 - Y_8 \cos Y_3 + Y_1(\cos Y_3)Y_9 + Y_7 \sin Y_3\big] \\
& + Y_{11}\big[(h - Y_2)\cos Y_3 + Y_1 \sin Y_3\big] \\
& + 2Y_4^2\big[(h - Y_2)(\cos Y_3)Y_9 - Y_8 \sin Y_3 + Y_1(\sin Y_3)Y_9 - Y_7 \cos Y_3\big] \\
& + 4Y_4Y_{10}\big[(h - Y_2)\sin Y_3 - Y_1 \cos Y_3\big]\Big\}.
\end{aligned}
\tag{28}
$$

For example, integrating this system with $i = 1$ up to $\eta = \pi$ gives $Y_7(\pi) = \partial x(\pi; v)/\partial v_1$.

However, the constant h must satisfy

$$
h = y(\pi; v) - \frac{\theta''(\pi; v)}{\pi B \alpha},
\tag{29}
$$

which is simply a nonlinear scalar equation for h. So, the overall algorithm is:

1) Fix a value for h.
2) Solve $F(v) = 0$ for v (keeping h fixed).
3) Compute $y(\pi; v)$ and $\theta''(\pi; v)$.
4) If Eq. (29) is satisfied, stop. Otherwise, adjust the value of h using the secant method and return to step 2).

It would have been possible to formulate the entire problem as a single nonlinear system in the three unknowns v_1, v_2, and h, and simultaneously solve for v and h. The above decoupling is more efficient computationally, however.

Discussion of Results

Figure 2 shows the curvatures or normalized moments. The maximum moment occurs at $s = \pi$ or at the location of the point mass. Of much smaller magnitude is the moment at $s = 0$. Both moments increase with α and B. Our approximate formulas compare well with exact numerical integration for $B < 1$. Figure 3 shows the maximum length b increases with α and B while the maximum width a decreases. Both Figures 2 and 3 are useful in the design of freely rotating rings. Figures 4 and 5 show the deformed configurations of the ring for some values of α and B. We see the shape is neither circular nor elliptic, but egg-shaped.

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Figure Captions

Figure 1. The coordinate system.

Figure 2. The curvature or normalized moment as a function of B for various α. ——— $\theta'(\pi)$; — · — · — $\theta'(0)$; — — — — approximations Eq. (19) or Eq. (20).

Figure 3. The maximum length b and the maximum width a. — · — · — distance h to center of mass; — — — — approximations Eq. (21) or Eq. (23).

Figure 4. Configurations for the same rotation rate $B = 2$. From left to right, $\alpha = 0.25$, $\alpha = 0.5$, $\alpha = 1$. Cross indicates center of mass.

Figure 5. Configurations for the same mass $\alpha = 2$. From left to right, $B = 1$, $B = 2$, $B = 3$. Cross indicates center of mass.

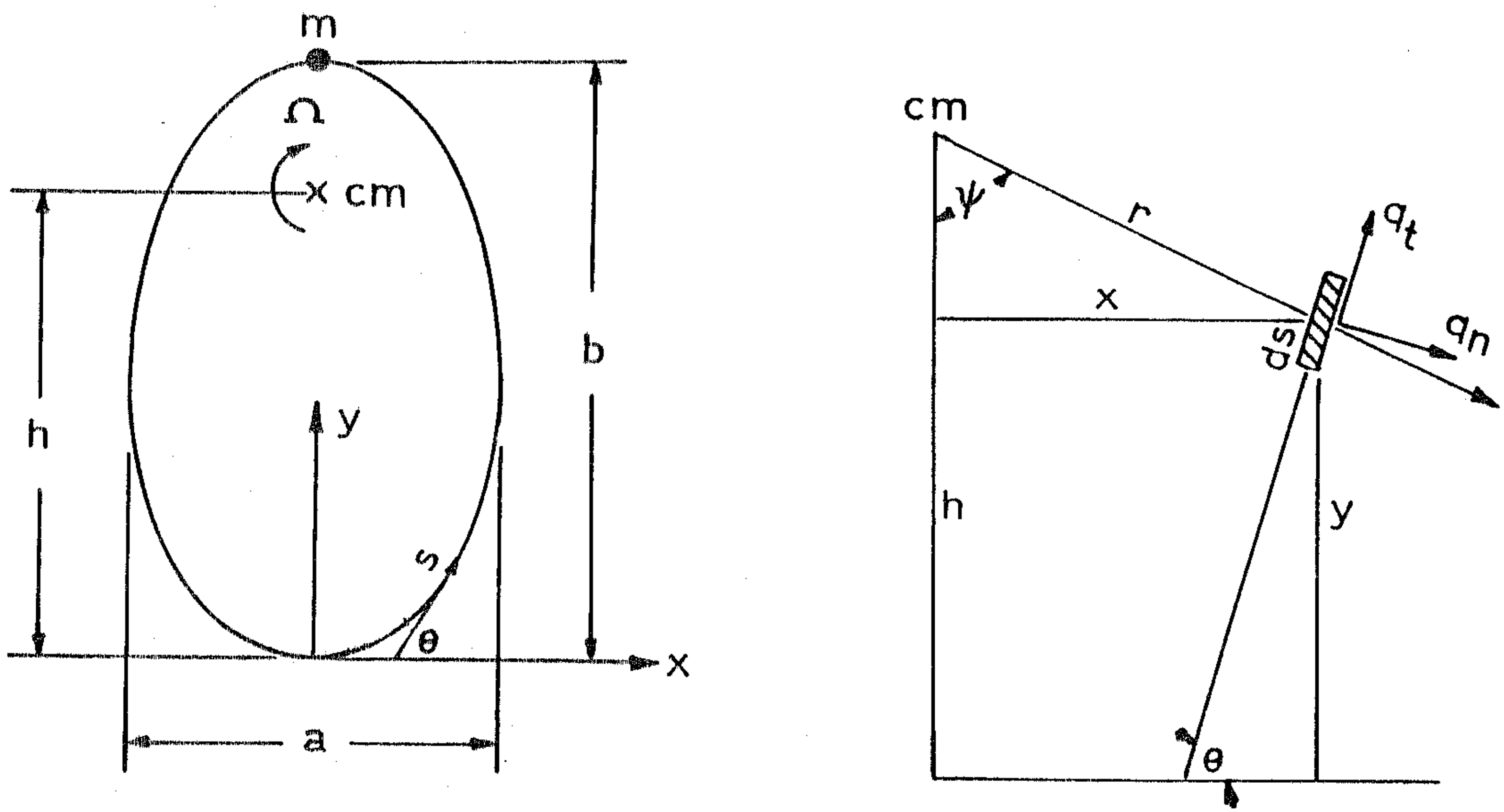

Fig 1

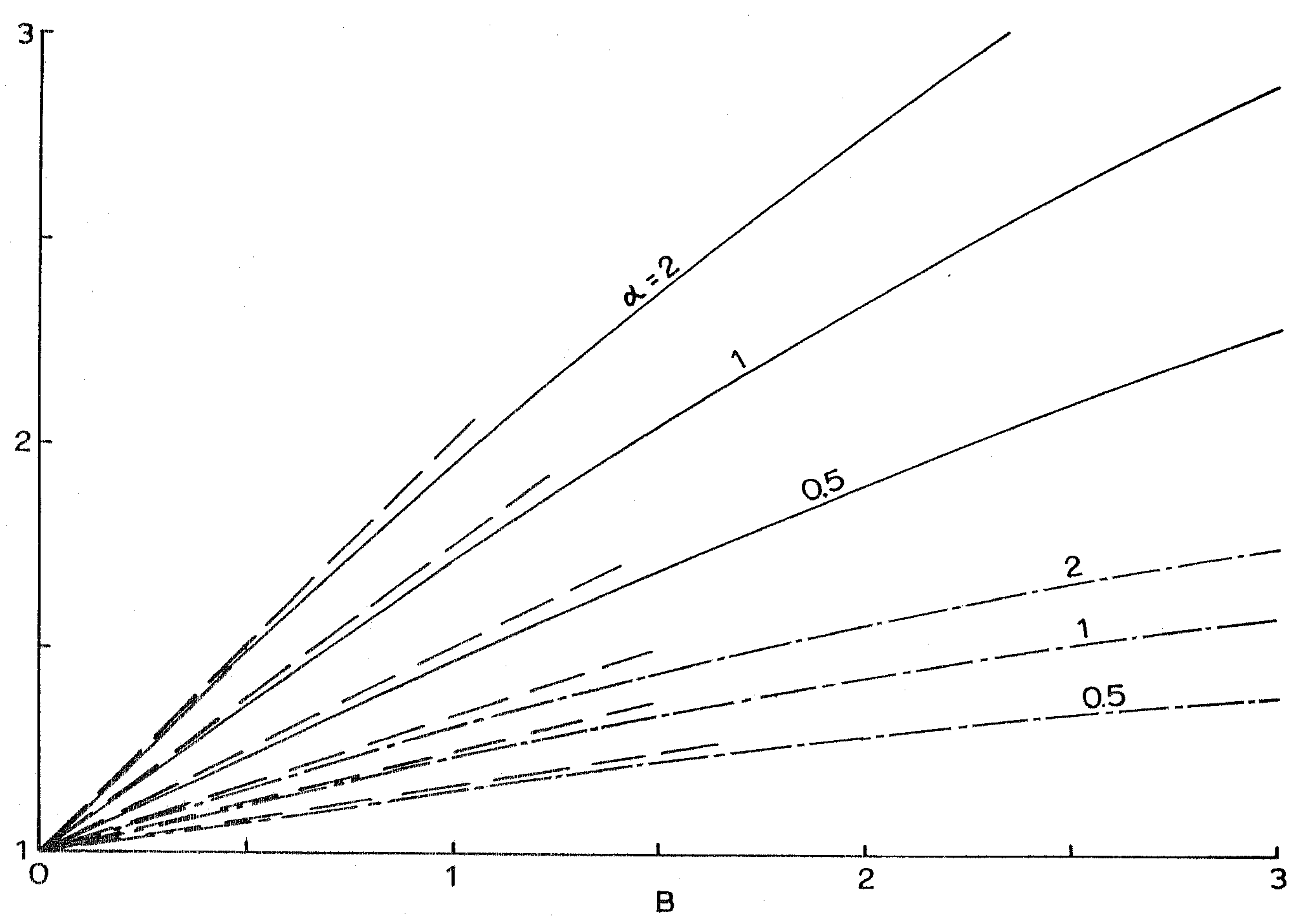

Fig 2

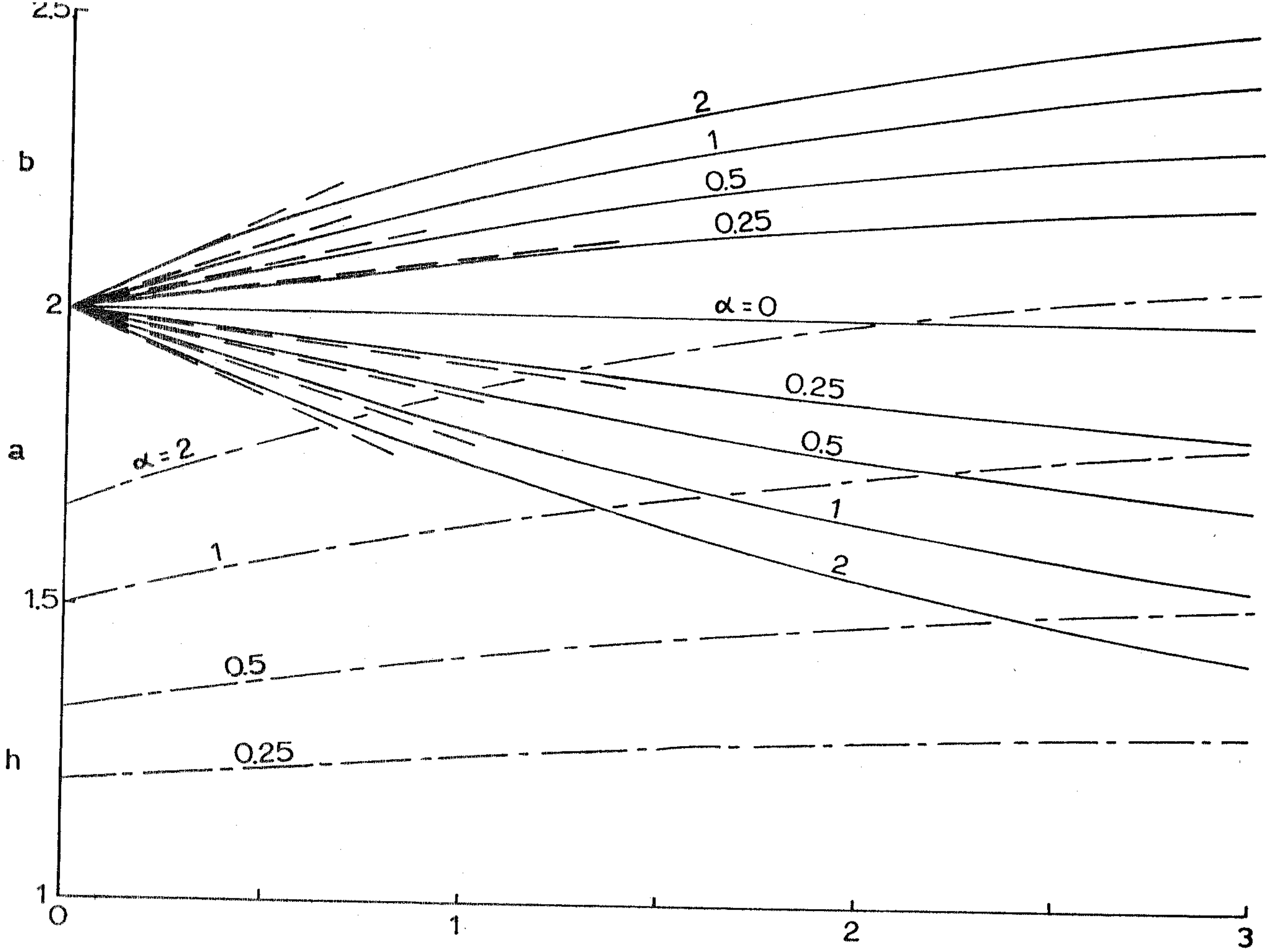

Fig 3

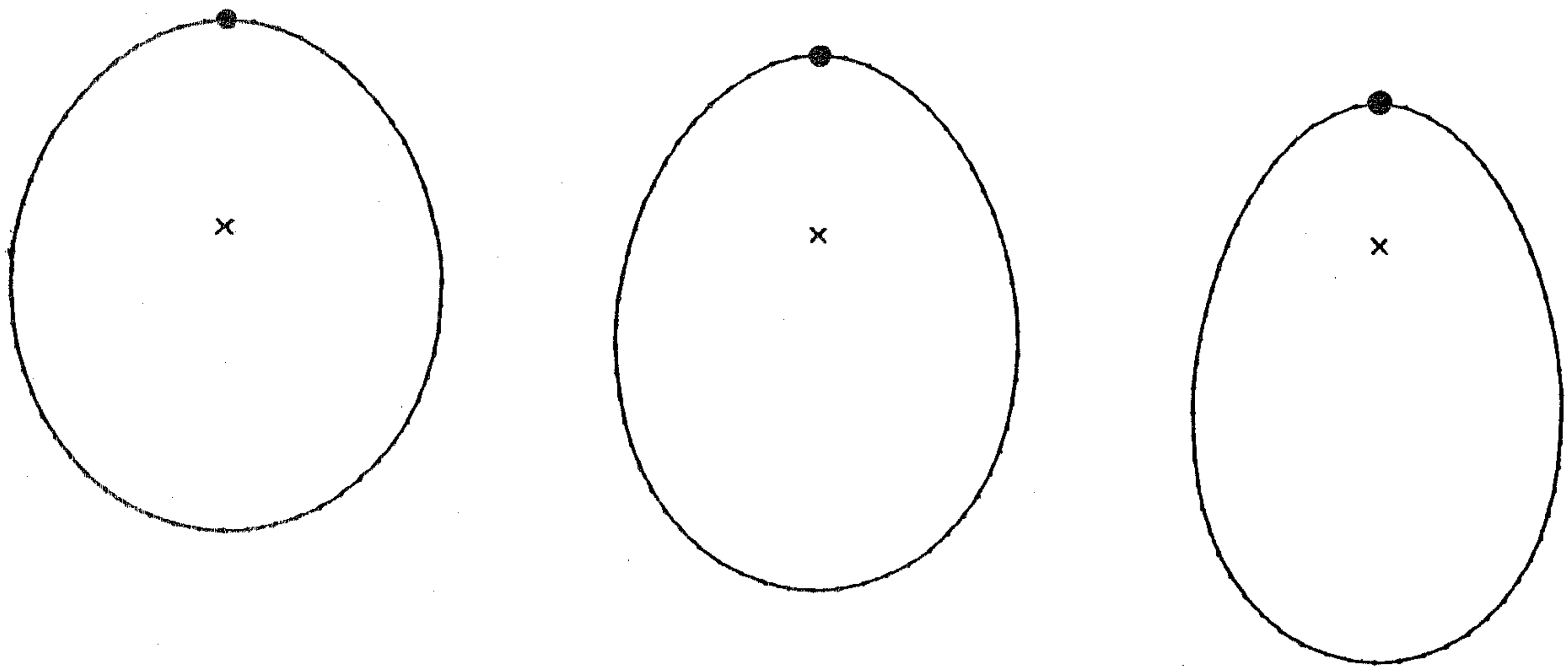

Fig 4

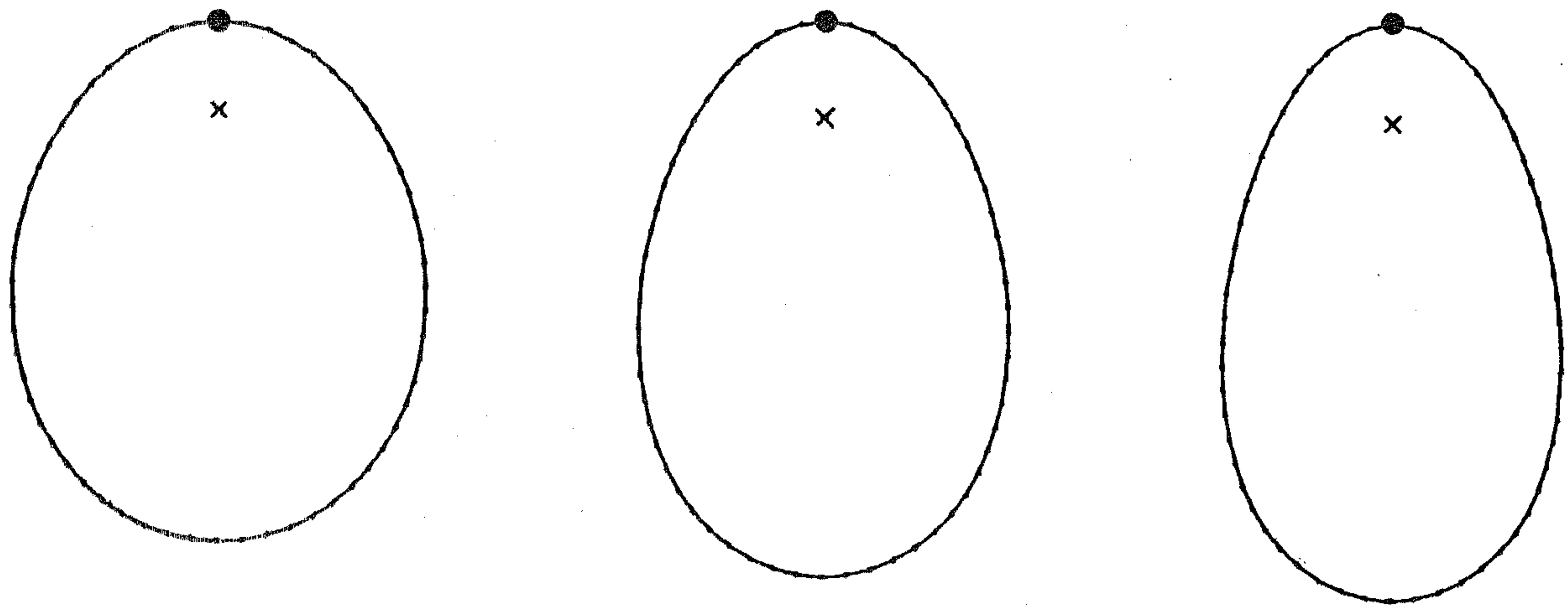

Fig 5