CP-Rays in Simplicial Cones

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CP-RAYS IN SIMPLICIAL CONES

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ABSTRACT:

In classical mathematics, interest in the concept of regularity of a triangle, is mainly centered on the property that for every interior point of the triange, its orthogonal projection on the line containing each side must lie in the relative interior of that side. We generalize the concept of regularity using this property, and extend this work to simplicial cones in R^n, and derive very efficient necessary and sufficient conditions for this property to hold in them. We show that these concepts have important ramifications in algorithmic studies of the linear complementarity problem. We relate our results to other well known properties of square matrices.

KEY WORDS: Simplicial cones, faces, orthogonal projections, CP-points and rays, linear complementarity problem, Positive definite matrices, Z-matrices, P-matrices.

1. INTRODUCTION

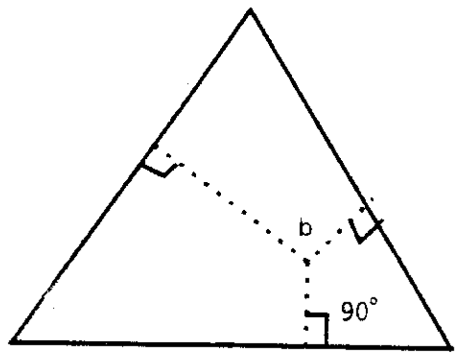

Figure 1: An equilateral triangle
and a CP-point b in it.

Consider the equilateral triangle in R^2 shown in Figure 1. The point b in Figure 1 satisfies the following properties.

i) it is in the interior of the triangle

ii) for each side of the triangle, the orthogonal projection of b on the straight line containing that side, is in the relative interior of that side.

We will call a point satisfying these two properties, a CP-point (abbreviation for "centrally (or interiorly) projecting point", since it projects into the interior of each side) for the triangle. See Figure 2 for a triangle in R^2 and a point c in it which is not a CP-point, since it violates (ii).

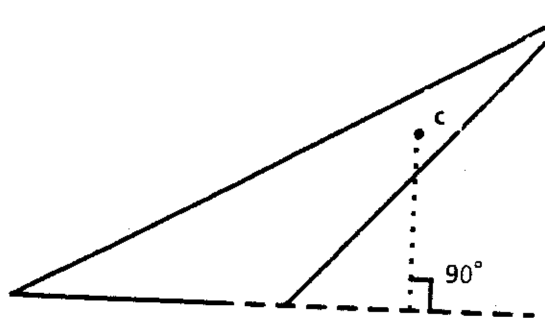

Figure 2: A triangle in R^2 and a point c in it which is not a CP-point.

Every triangle in R^2 has a CP-point. By a well-known result in classical geometry, the bisector lines of the three angles of a triangle have a common point, b, and clearly that point b is a CP-point for the triangle. See Figure 3.

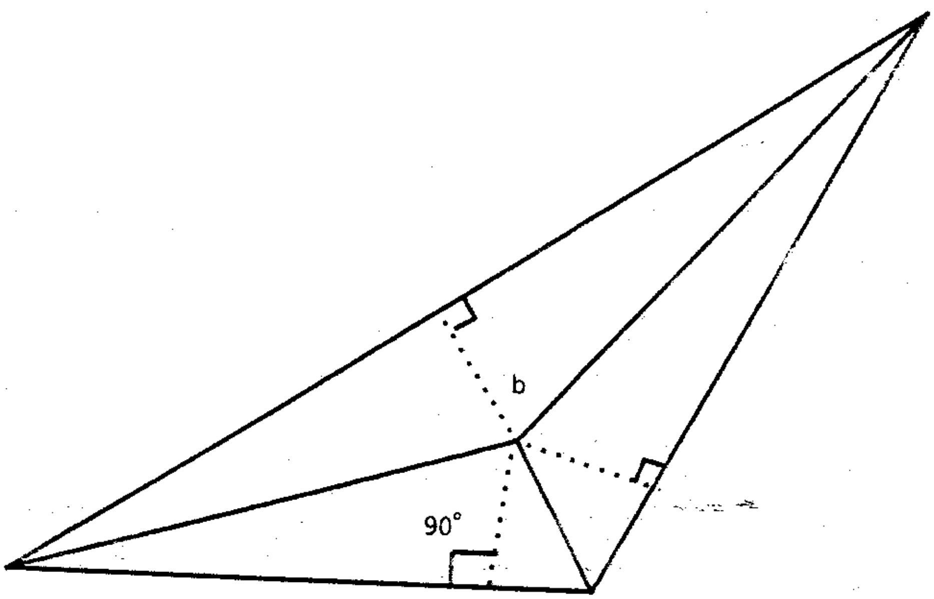

Figure 3: The point b where all the bisectors lines of the angles in a triangle meet, is a CP-point for it.

Also, it can be verified that every interior point of the equilateral triangle in Figure 1 is a CP-point for it, but this property does not hold for the triangles in Figures 2 and 3.

In this paper we generalize the concept of CP-points for triangles in R^2 to simplicial cones in R^n. We show that CP-points in simplicial cones play an important role in studies aimed towards developing efficient algorithms for linear complementarity problems associated with positive definite symmetric matrices and P-matrices. We investigate several geometric properties associated with CP-points in simplicial cones, and relate them to classical results in matrix theory.

2. NOTATION

We will use the following notation.

LCP	Linear Complementarity problem, defined in Section 4.
LCP(q, M)	an LCP for which the input data is the column vector q and square matrix M.
$E_{i.}$, $E_{.j}$	If E is any matrix, $E_{i.}$ denotes its ith row vector, $E_{.j}$ denotes its jth column vector.
E_{LJ}	If $E = (e_{ij})$ is any matrix of order $m \times n$, given $L \subset \{1, \cdots, m\}$, $J \subset \{1, \cdots n\}$, E_{LJ} denotes the submatrix $(e_{ij} : i \in L, j \in J)$ of E determined by these subsets.
$D_{J.}$	submatrix of D with row vectors $D_{i.}$ for $i \in J$.
$D_{.J}$	Submatrix of D with Column vectors $D_{.j}$ for $j \in J$
x_J	If $x = (x_j) \in R^n$, $J \subset \{1, \cdots n\}$, x_J denotes the column vector $(x_j : j \in J)$
Pos(D)	given the matrix D, this is the cone $\{x : x = Dy \text{ for some } y \geqq 0\}$

LH (D)	the linear hull of the column vectors of the matrix D, it is the subspace $\{x: x = Dy$, for some vector $y\}$.
I	Unit matrix of order n.
$\|J\|$	Cardinality of the set J.
$K \backslash \Delta$	When K and Δ are two sets, $K \backslash \Delta$ is the set of elements of K which are not in Δ.
P-matrix	a square matrix, all of whose principal subdeterminants are >0.
Z-matrix	a square matrix, all of whose off-diagonal elements are $\leqq 0$.
M-matrix	a Z-matrix which is also a P-matrix
P_n, Z_n, M_n	these are respectively the classes of P – , Z – . and M – matrices of order n.
$\|\|x\|\|$	The Euclidean norm of the vector $x = (x_j)$, it is $+\sqrt{\sum_j x_j^2}$
Γ	the set $\{1, \cdots, n\}$
ray of a point in R^n	for $b \in R^n$, $b = 0$, the ray of b is $\mathrm{Pos}(b) = \{x : x = \alpha b$ for some $\alpha \geqq 0\}$.
π	pi, the length of the circumference of a circle in R^2 with diameter 1.
PD	positive definite

3. CP-POINTS AND CP-RAYS IN SIMPLICIAL CONES

Let D be a real nonsingular square matrix of order n. Pos(D) is a simplicial cone in R^n. For each j = 1 to n, the ray $\mathrm{Pos}(D_{.j}) = \{x: x = \alpha D_{.j}, \alpha \geqq 0\}$ is a generator or a generator ray for Pos(D). In this section we define CP-points and CP-rays for Pos(D), and various other geometric entities related to Pos(D), which are used later in studying CP-points.

Since Pos(D) is a simplicial cone, a face of Pos(D) is Pos($D_{.J}$) for some $J \subset \Gamma$ and vice versa. For any $J \subset \Gamma$, we will say that Pos($D_{.J}$) is the face of Pos(D) corresponding to the subset J. When $|J| = n - 1$, Pos ($D_{.J}$) is called a *facet* of Pos(D), it is a face of Pos(D) whose dimension is one less than the dimension of Pos(D).

The point b $\in$Pos(D) is said to be a CP-point for Pos(D) if it satisfies the following properties 1,2.

1. b is in the interior of Pos(D)

2. for every face F of Pos(D), the orthogonal projection of b on the linear hull of F, is in the relative interior of F.

Since there are $2^n - 2$ nonempty proper faces of Pos(D), property 2 consists of $2^n - 2$ conditions. Here again, "CP-point" is an abbreviation for "centrally (or interiorly) projecting point", since this point projects into the interior of every face of Pos(D).

As an example, consider the simplicial cone, Pos(D) in R^2, given in Figure 4. Pos(D) is an angle and its interior in the two dimensional Cartesian plane, in this case an obtuse angle. The point $(1, 1/2)^T$ violates property 2, since its orthogonal projection on the linear hull of Pos($D_{.1}$) is d, which is not even in Pos($D_{.1}$). The point $(0, 1)^T$ also violates property 2, since its orthogonal projection on the linear hull of Pos($D_{.2}$) is 0, which is not in the relative interior of Pos($D_{.2}$). However, the point $b = (-1 + \sqrt{2}, 1)^T$ on the bisector line of the angle Pos(D) does satisfy both properties 1,2, and is therefore a CP-point for Pos(D) in this example.

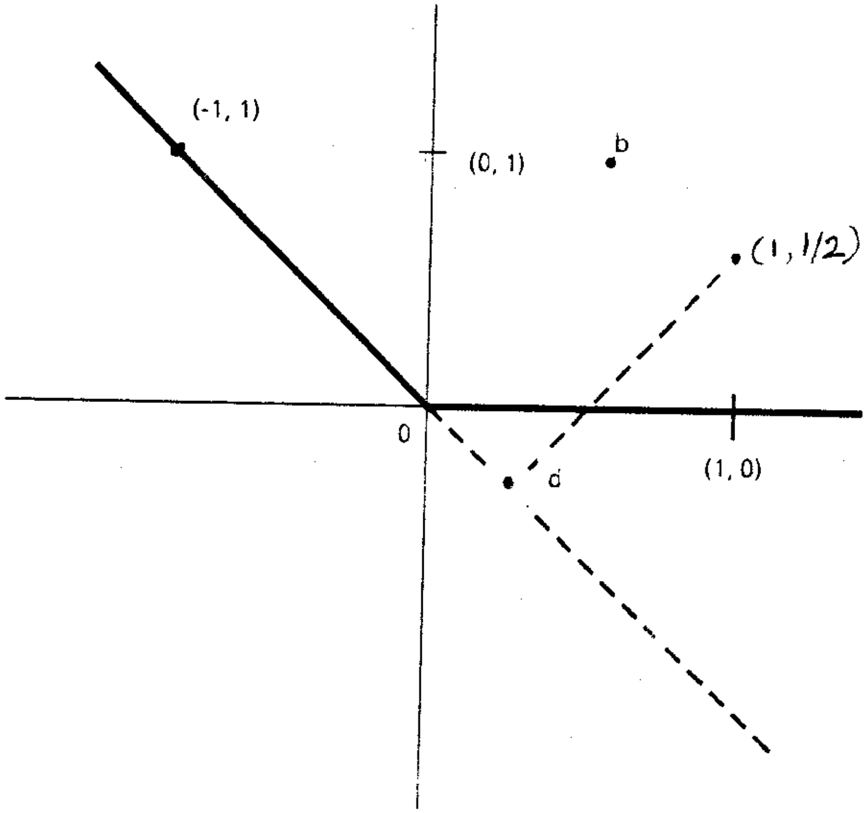

Figure 4: $D = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. Pos (D) is the cone marked by the angle sign bounded by thick lines. The point b on the bisector line of this angle, is a CP-point for Pos (D).

If b is a CP-point for Pos (D), the ray Pos (b) = $\{x: x = \alpha b, \text{ for } \alpha \geqq 0\}$ is said to be a CP-ray for Pos (D).

LEMMA 1: Every nonzero point on a CP-ray for Pos (D) is a CP-point for it.

PROOF: Let b be a CP-point for Pos (D). Since b satisfies property 1, so does αb for $\alpha > 0$.

Let F be a face of Pos(D) and $\bar{x}$, the orthogonal projection of b on the linear hull of F. Then, clearly, the orthogonal projection of α b on the linear hull of F is $\alpha \bar{x}$, for any $\alpha > 0$. Since b satisfies property 2, this implies that α b also satisfies property 2 for any $\alpha > 0$. So, α b is a CP-point for Pos(D) for every $\alpha > 0$.

PROJECTION AND NON-PROJECTION FACES OF POS (D) RELATIVE TO A GIVEN POINT b.

Given $b \in R^n$, the face Pos $(D_{.J})$ of Pos (D) corresponding to $J \subset \Gamma$, is said to be a projection face relative to b if the orthogonal projection of b in $LH(D_{.J})$ is in the relative interior of Pos $(D_{.J})$; non-projection face relative to b otherwise.

The face of Pos (D) corresponding to Γ is Pos (D) itself. By the above definition, Pos (D) is a projection face relative to b iff b is in the interior of Pos (D).

Thus b is a CP-point iff every face of Pos (D) is a projection face relative to b, that is, iff there are no non-projection faces relative to b.

The concept of a projection face is algorithmically very important in the study of the linear complementarity problem. In K. G. Murty and Y. Fathi [13] and P. Wolfe [18] it has been used to develop efficient algorithms for nearest point problems in simplicial cones, and special types of linear complementarity problems. These algorithms, and consequently projection faces, play an important role in a new algorithm for linear programming developed by S. Y. Chang and K. G. Murty [1].

THE SET OF CP-POINTS OF POS (D) WHEN n = 2.

If n = 2, and Pos (D) is an acute or right angle, every point in the interior of Pos (D) is a CP-point for it. If Pos (D) is an obtuse angle, any point b in the interior of Pos $((D^T)^{-1})$ (these are points b satisfying $D^T b > 0$) is a CP-point for Pos (D). See Figures 5, 6.

In this case (n = 2), the besector ray of the angle Pos (D), is always a CP-ray for Pos (D). A nonzero point on this bisector ray is

$$b = \frac{1}{2}\left(\frac{D_{.1}}{||D_{.1}||} + \frac{D_{.2}}{||D_{.2}||}\right),$$

the bisector ray is the ray of this point b.

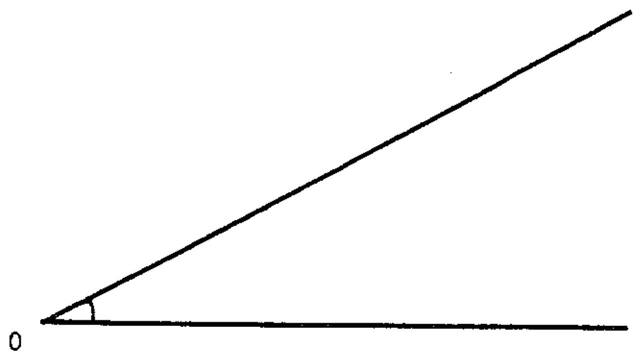

Figure 5: When D is of order 2, and Pos (D) is an acute or right angle, every interior point of Pos (D) is a CP-point for Pos (D).

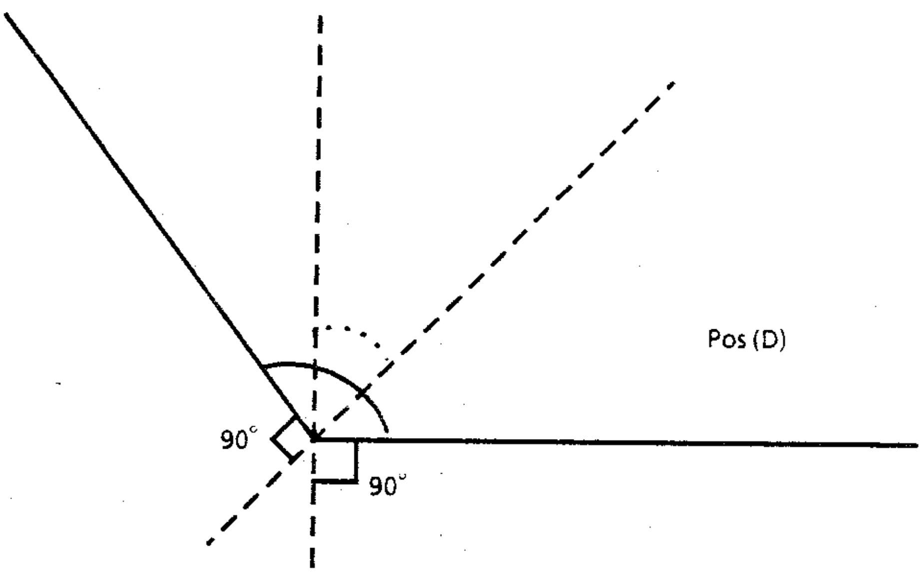

Figure 6: When D is of order 2, and Pos (D) is an obtuse angle, $\{b: D^T b > 0\}$, the open cone bounded by the dashed lines, is the set of all CP-points for Pos (D).

INCENTER AND CIRCUMCENTER OF POS (D).

Facets of Pos (D) are its faces of dimension $n-1$. For $j = 1$ to n, let $K_j = \text{Pos}(D_{.1}, \cdots, D_{.j-1}, D_{.j+1}, \cdots, D_{.n})$ and let H_j be the linear hull of K_j. $K_1, \ldots, K_n$ are the facets and $H_1, \cdots, H_n$ are the facetal hyperplanes for Pos (D).

The concept of the bisector ray of the angle Pos (D) when $n = 2$, does not directly generalize for $n \geqq 3$. However, when $n = 2$, the important property of a nonzero point on the bisector ray of Pos (D) is that it is equidistant from each facet of Pos (D), and this property can be generalized for $n \geqq 3$. There is a point in the interior of Pos (D) which is equidistant (say, at a distance of 1) from each of the facetal hyperplanes for Pos (D), this point is called the incenter for Pos (D) and its ray is called the incenter ray for Pos (D). Each point on the incenter ray is equidistant from each of the facetal hyperplanes. See Figure 7.

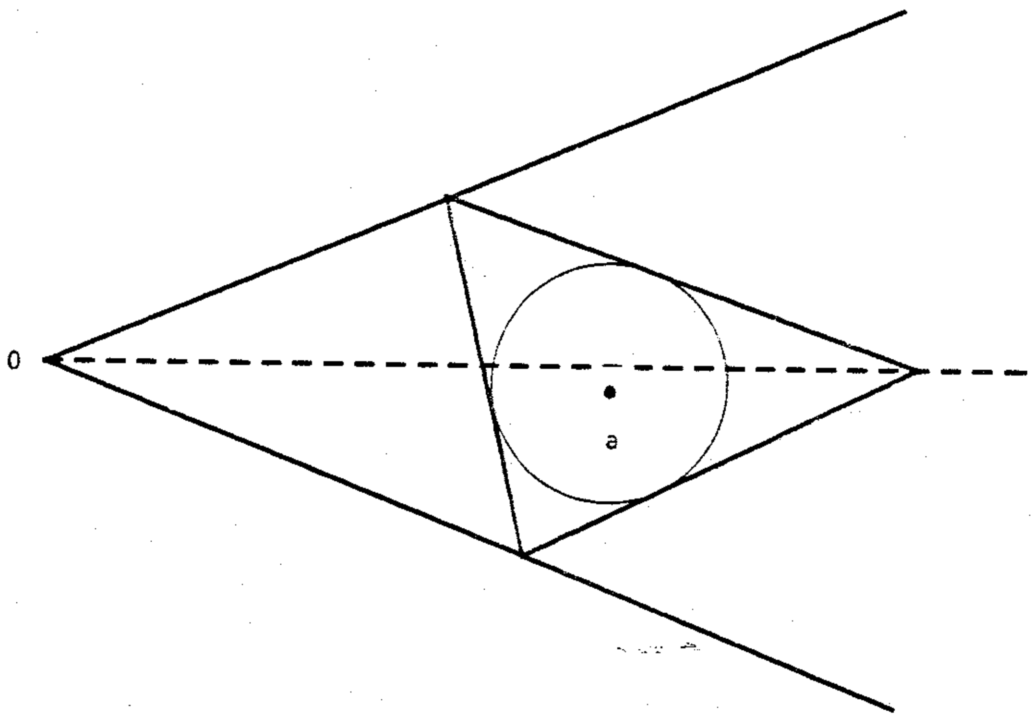

Figure 7: A simplicial cone in R^3 and its incenter a.

Let $\beta = (\beta_{ij}) = D^{-1}$. Then, for $j = 1$ to n, the facetal hyperplane H_j for Pos (D) is

$$H_j = \{x : \beta_{j.} x = 0\}. \qquad (1)$$

The incenter of Pos (D) which is at a distance of 1 from each facetal hyperplane of Pos (D) is a, where

$$a = D(\delta_1, \ldots, \delta_n)^T \tag{2}$$

where $\delta_j = \|\beta_{j\cdot}\|$. This point a is clearly in the interior of Pos (D), and the ray of a is the incenter ray for Pos (D).

When n = 2, the incenter ray for Pos (D) is exactly its bisector ray, and it is a CP-ray. One may be tempted to conjecture that the incenter ray is always a CP-ray for Pos (D). Unfortunately, this conjecture may be false when $n \geqq 3$. Let

$$D = \begin{pmatrix} -1 & 1 & 20 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

When D is the matrix in (3), the incenter ray is not a CP-ray for Pos (D). Surprisingly, we found that when D is the matrix given in (3), Pos (D) has no CP-point at all. See Section 10 for a geometric argument to establish this fact.

At this stage we were quite tempted to conjecture that if D is a square nonsingular matrix such that the incenter ray is not a CP-ray for Pos (D), then Pos (D) has no CP-points at all. Unfortunately, this conjecture also turned out to be false. Let

$$D = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

When D is the matrix given in (4), the incenter is $a = (-1, 1+\sqrt{2}, 1)^T$, and it is not a CP-point for Pos (D). But $b = (2, 8, 1)^T$ is a CP-point for Pos (D).

Given the nonsingular square matrix D, the point which is equidistant (say, at a distance of 1) from each of the generator rays of Pos (D) is known as the circumcenter for Pos (D), and the ray of the circumcenter is known as the circumcenter ray for Pos (D). Se Figures 8, 9. While the incenter is always contained in the interior of Pos (D), the circumcenter may not even be in Pos (D).

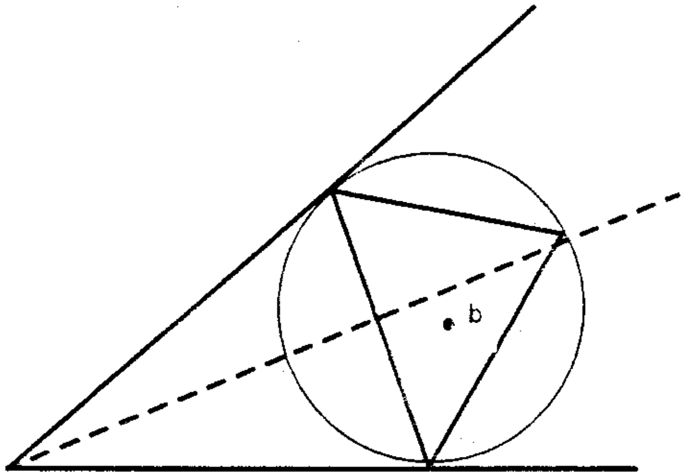

Figure 8: A simplicial cone in R^3 with its circumcenter b which is in the cone in this case.

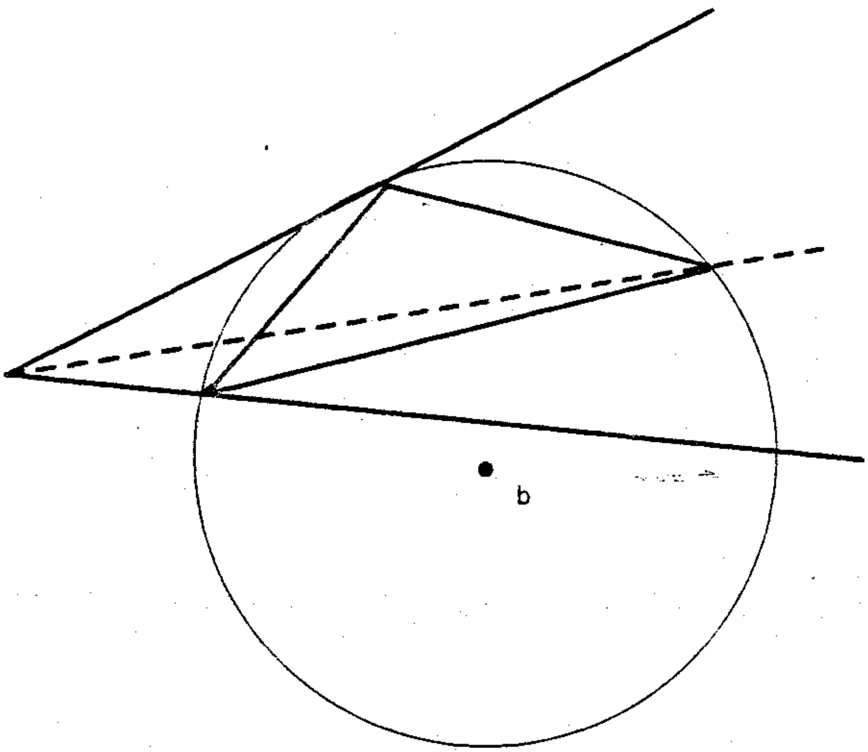

Figure 9: A simplicial cone in R^3 for which the circumcenter b is outside the cone.

The circumcenter ray makes equal angles with all the generator rays of Pos (D) (this provides another equivalent definition of the circumcenter ray for a simplicial cone). From this it can be verified that the circumcenter ray for Pos (D) is the ray of $(D^T)^{-1}\tau$ where $\tau = (\tau_1, \cdots, \tau_n)^T$, $\tau_j = ||D_{.j}||$, $j = 1$ to n. So the circumcenter ray is in the cone Pos (D) iff $(D^TD)^{-1}\tau \gneqq 0$.

POLAR CONE

Let D be a real nonsingular square matrix of order n. The cone

$$\{y: \quad y^Tx \geqq 0 \text{ for every } x \in \text{Pos}(D)\}$$

is known as the Polar Cone of the cone Pos (D). Clearly, y is contained in the polar cone of Pos (D) iff

$$y^T D_{.j} \geqq 0, \text{ for all } j = 1 \text{ to } n$$

i.e., $D^Ty \geqq 0$.

This implies that the polar cone of Pos (D) is Pos ($(D^T)^{-1}$).

The Gale-Nikaido theorem ([4]) states that if A is a P-matrix, then the system $Ax \leqq 0$, $x \geqq 0$, has the unique solution $x = 0$. Using this and the Gordan's theorem of the alternatives ([7, 11]), since $(D^TD)^{-1}$ is a PD matrix and hence a P-matrix, we conclude that the system

$$\begin{aligned} D^{-1}\,(D^T)^{-1}x &> 0 \\ x &> 0 \end{aligned}$$

has a solution $\bar{x}$. This implies that the point $(D^T)^{-1}\bar{x}$ is both in the interior of Pos (D) and the interior of its polar cone. Thus, the interior of Pos (D) and its polar cone always have a nonempty intersection. We will prove later (Corollary 1) that every CP-point for Pos (D) must be an interior point of its Polar Cone Pos ($(D^T)^{-1}$).

Here are some important properties of the polar cone of Pos (D). The circumcenter ray for Pos (D) can be verified to lie always in the polar cone of Pos (D).

Let $\beta = D^{-1}$ and let $H_1 = \{x : \beta_{1.}\, x = 0\}$ be the facetal hyperplane containing Pos $(D_{.2}, \cdots, D_{.n})$. If $y \notin H_1$, the orthogonal projection of y on H_1 is

$$\hat{y} = y - \frac{(\beta_{1.})^T (\beta_{1.}) y}{\|\beta_{1.}\|^2}$$

Since $\beta_{1.} D_{.j} = 0$ for all $j = 2$ to n, it can be verified that if $y^T D \geqq 0$, then $(\hat{y})^T D_{.j} \geqq 0$ also, for all $j = 2$ to n. This implies that the orthogonal projection of the polar cone of Pos (D) on $H_1 = LH\{D_{.2}, \cdots, D_{.n}\}$ is the polar cone of the face Pos $(D_{.2}, \cdots, D_{.n})$. In the same way it can be verified that the orthogonal projection of the polar cone of Pos (D) on the linear hull of any face of Pos (D) is the polar cone of that face.

DIHEDRAL ANGLES AND INWARD NORMALS TO FACETAL HYPERPLANES

As before, let D be a square nonsingular matrix of order n and $\beta = D^{-1}$. The hyperplane H_j defined in (1) are the facetal hyperplanes for Pos (D). For i, $j \in \Gamma$, $i \neq j$, the solid angle between the facetal hyperplanes H_i and H_j is known as a dihedral angle.

Let $G_{.j} = (\beta_{j.})^T / \|\beta_{j.}\|$. The ray of $G_{.j}$ is normal to the facetal hyperplane H_j, and is on the same side of H_j as Pos (D), hence it is known as an inward normal to the facetal hyperplane H_j.

For $i, j \in \Gamma$, $i \neq j$, a measure of the dihedral angle between H_i, H_j in radians is

$\theta = \pi -$ (angle between inward normals to H_i, H_j in radians).

$= \pi - \mathrm{Cos}^{-1} ((G_{.i})^T G_{.j})$

where $\mathrm{Cos}^{-1}(\cdot)$ is the inverse cosine function in radians.

Thus there are $\binom{n}{2}$ dihedral angles associated with any simplicial cone in R^n.

The polar cone of Pos (D), Pos ($(D^T)^{-1}$), is a subset of Pos (D) iff, for every $y \geqq 0$, $(D^T)^{-1}y = Dx$ for some $x \geqq 0$, that is, iff $D^{-1}(D^T)^{-1} y \geqq 0$ for all $y \geqq 0$. This holds iff $D^{-1}(D^T)^{-1} = \beta\beta^T \geqq 0$, that is, iff all the dihedral angles associated with Pos (D) are non-acute (i.e., obtuse or right). Similarly, it can be verified that the polar cone of Pos (D) is in the interior of Pos (D) iff all the dihedral angles associated with Pos (D) are strictly obtuse, and in this case the circumcenter ray of Pos (D) is in its interior.

CP-OWNING, OR CP-LACKING SIMPLICIAL CONES

Given the nonsingular square matrix D of order n, we will say that the simplicial cone Pos (D) is a CP-owning (or CP-owner) simplicial cone, if it has at least one CP-point, or a CP-lacking (or CP-lacker) simplicial cone otherwise.

We have already shown that every two dimensional simplicial cone is a CP-owner. But for $n \geqq 3$, simplicial cones may or may not be CP-owners.

CP-POINTS FOR LOWER DIMENSIONAL CONES

Let A be an $n \times r$ matrix where $r < n$, whose set of column vectors is linearly independent. Then Pos (A) is an r-dimensional cone in R^n which has empty interior. A CP-point for Pos (A) is defined exactly as before, with the exception that property 1 now requires the point to be in the relative interior of Pos (A).

4. APPLICATION TO LCP

Let $M = (m_{ij})$ be a given real square matrix of order n and $q = (q_j)$ a given column vector in R^n. The LCP with data q, M, denoted by (q, M), is the problem of finding vectors $w = (w_j)$, $z = (z_j) \epsilon R^n$, satisfying

$$\begin{aligned} & w - Mz = q \\ & w, z \geqq 0 \\ & w^T z = 0 \end{aligned} \tag{5}$$

The LCP is a fundamental problem in mathematical programming, it has been the focus of extensive research over the last 25 years. CP-points came up in a study dealing with the construction of efficient pivotal algorithms for solving certain classes of LCPs. In this section we review this application of CP-points in LCP.

In (5), the pair of variables $\{w_j, x_j\}$ is known as the jth complementary pair of variables, for $j = 1, \ldots, n$. In (5), the column vector associated with w_j is $I_{.j}$, and the column vector associated with z_j is $-M_{.j}$. The pair $\{I_{.j}, -M_{.j}\}$ is the jth complementary pair of column vectors in (5). A complementary vector of variables in (5) is a vector $y = (y_1, \ldots, y_n)^T$ where $y_j \in \{w_j, z_j\}$ for each $j = 1, \ldots, n$. If $A_{.j}$ is the column in (5) associated with y_j, the matrix $A = (A_{.1} \ldots A_{.n})$ is known as the complementary matrix associated with y. The complementary vector y is said to be a complementary basic vector in (5) if the corresponding complementary matrix is nonsingular. If y is a complementary basic vector associated with the complementary matrix A, the complementary basic solution of (5) corresponding to y is given by

all variables in (5) not in y are $= 0$

$$y = A^{-1} q. \tag{6}$$

Clearly, the solution in (6) satisfies the complementarity condition $w^T z = 0$, because y is a complementary vector. If $A^{-1}q \geqq 0$, the solution in (6) is a solution of the LCP (q, M) (it satisfies all the conditions in (5)). It is then said to be a complementary basic feasible solution, and y is said to be a complementary feasible basic vector for (5). The complementary basic feasible solution in (6), and complementary feasible basic vector y are nondegenerate if $A^{-1}q > 0$.

For any $J \subset \Gamma$, the LCP of order $|J|$,

$$w_J - M_{JJ} z_J = q_J$$

$$w_J, z_J \geqq 0, \quad (w_J)^T z_J = 0 \tag{7}$$

which is the LCP (q_J, M_{JJ}), is known as the principle subproblem of (5) corresponding to the subset J.

One method for solving the LCP (q, M) when M is a P-matrix is the following. Select a column vector $p>0$ in R^n. Consider the following parametric LCP, where α is a nonnegative parameter

$$
\begin{array}{cc|c}
w & z & \\
\hline
I & -M & q+\alpha p \\
\hline
\multicolumn{2}{c|}{w, z \geqq 0,\ w^T z = 0} &
\end{array}
\tag{8}
$$

When $\alpha>0$ is sufficiently large, w is a complementary feasible basic vector for (8), and the method is initiated with this. The method moves through complementary basic vectors for (8), exchanging one basic variable by its complement in each pivot step, in an effort to find a complementary feasible basic vector for (8) for smaller and smaller values of α until it reaches the value 0. In some general stage, let $(w_{\bar{J}}, z_J)$ be the complementary basic vector for (4) at this stage, for some $J \subset \Gamma$, $\bar{J} = \Gamma \setminus J$. The corresponding basic solution for (8) is

$$(w_J, z_{\bar{J}}) = 0$$

$$\begin{pmatrix} z_J \\ w_{\bar{J}} \end{pmatrix} = \begin{pmatrix} \bar{q}_J + \alpha\bar{p}_J \\ \bar{q}_{\bar{J}} + \alpha\bar{p}_{\bar{J}} \end{pmatrix} \tag{9}$$

where

$$(\bar{q}_J, \bar{p}_J) = -(M_{JJ})^{-1}(q_J, p_J),$$
$$(\bar{q}_{\bar{J}}, \bar{p}_{\bar{J}}) = (q_{\bar{J}}, p_{\bar{J}}) + M_{\bar{J}J}(\bar{q}_J, \bar{p}_J).$$

The smallest value of α for which the right hand side of (9) remains $\geqq 0$ is θ given by

$$\theta = \begin{cases} \text{Maximum } \{-\bar{q}_j/\bar{p}_j : j \text{ such that } \bar{p}_j > 0\} \\ -\infty, \text{ if } \bar{p} \leqq 0 \end{cases} \tag{10}$$

If $\theta \leqq 0$, $(w_J, z_{\bar{J}})$ is a complementary feasible basic vector for (8) when $\alpha = 0$, that is, a complementary feasible basic vector for (5), terminate. Otherwise, let r be the maximizing index in (10) (or the maximum among these indices, if there is a tie). Perform a pivot step replacing the basic variable from the pair $\{w_r, z_r\}$ in the current complementary basic vector by its complement, and continue the method with the new complementary basic vector.

This method is known to solve the LCP (q, M), when M is a P-matrix, in a finite number of pivot steps [8]. Even though this method is finite, in the worst case, it may take up to 2^n pivot steps before termination [9, 11]. However, in [14] J. S. Pang and R. Chandrasekaran observed that if the column vector p is selected so that it satisfies the following properties 3, 4, then the method will terminate after at most n pivot steps, for any $q \epsilon R^n$.

3. $p > 0$
4. $(M_{JJ})^{-1} p_J > 0$, for all nonempty $J \subset \Gamma$.

If the column vector p satisfies properties 3 and 4, in the above method, when $(w_{\bar{J}}, z_J)$ is the complementary basic vector, in the basic solution (9), $\bar{p}_j < 0$ for all $j \epsilon J$; and the maximizing index r giving the value of θ in (10) will not be in J. That is, once a z-variable becomes basic in the above method, it will stay basic until termination. This guarantees that the method terminates after at most n pivot steps.

PC-VECTORS FOR A P-MATRIX

Given a P-matrix M of order n, the column vector $p \epsilon R^n$ is said to be a PC-vector for M if it satisfies properties 3 and 4 given above. Such a column vector was first defined in the Pang and Chandrasekaran paper [14], and hence the name "PC-vector".

It should be noted that property 4 stated above, actually involves $2^n - 1$ sets of conditions, one for each nonempty subset $J \subset \Gamma$.

The following Lemmas 2, 3 and Theorem 1 relating PC-vectors to CP-points follow from the results established in K. G. Murty [10].

LEMMA 2: Let $M \epsilon P_n$, $p \epsilon R^n$, $p > 0$. The column vector p is a PC-vector for M iff the vector composed of the z-variables only, is a nondegenerate complementary feasible basic vector in every principal subproblem of the LCP (11).

$$w - Mz = -p$$
$$w, z \geqq 0$$
$$w^T z = 0. \tag{11}$$

PROOF: Follows by direct verification.

□

We will now show that for the class of PD symmetric matrices M, PC-vectors correspond to CP-points of a related simplicial cone.

Let M be a real PD symmetric matrix of order n. It is well known [11, 12] that there exists a real nonsingular square matrix D of order n such that $M = D^T D$. One choice for D is the Cholesky factor of M [12]. Let D denote a real nonsingular square matrix of order n satisfying $D^T D = M$. Let $p \in R^n$, $p > 0$, and

$$b = (D^T)^{-1} p \tag{12}$$

LEMMA 3: For $J \subset \Gamma$, Pos $(D_{.J})$ is a projection face of Pos (D) relative to b iff $(M_{JJ})^{-1} p_J > 0$, or equivalently, iff z_J is the nondegenerate complementary feasible basic vector for the principal subproblem of (11) corresponding to J.

PROOF: If $J = \varnothing$, the result holds by convention. So, assume $J \neq \varnothing$ and let $|J| = s$. The orthogonal projection of b in LH$(D_{.J})$ is $D_{.J} \lambda_J$, where λ_J is the optimum solution of the problem

$$\begin{aligned}&\text{minimize } (b - D_{.J}\lambda_J)^T (b - D_{.J} \lambda_J) \\ &\text{over } \lambda_J \in R^s\end{aligned}$$

which is

$$\bar{\lambda}_J = (M_{JJ})^{-1} (D_{.J})^T b \tag{13}$$

So, the orthogonal projection of b in LH$(D_{.J})$, is in the relative interior of the face Pos $(D_{.J})$, if $\bar{\lambda}_J$ given by (13) is > 0, that is, iff $(M_{JJ})^{-1} (D_{.J})^T \, b = (M_{JJ})^{-1} p_J > 0$, or equivalently, iff z_J is the nondegenerate complementary feasible basic vector for the principal subproblem of (11) corresponding to J.

□

THEOREM 1: Let D be a real nonsingular square matrix of order n and $p \in R^n$, $p > 0$. Let $M = D^T D$, $b = (D^T)^{-1} p$. The vector p is a PC-vector for M iff b is a CP-point for Pos (D).

PROOF: Follows directly from Lemmas 2, 3. □

COROLLARY 1: Let D be a real nonsingular square matrix of order n. Every CP-point for Pos (D) must be an interior point of the Polar Cone of Pos (D), Pos ($(D^T)^{-1}$).

PROOF: Let b be a CP-point for Pos (D). Then by Theorem 1, $p = D^T b$ is a PC-vector from $M = D^T D$, and hence $p = D^T b > 0$. Since $b = (D^T)^{-1} p$, this implies that b is an interior point of Pos ($(D^T)^{-1}$). □

Corollary 1 can also be proved very directly. If b is a CP-point for Pos (D), each of the generator rays Pos ($D_{.j}$), $j = 1$ to n, must be a projection face relative to b, by definition. This implies that $(D_{.j})^T b > 0$ for all $j = 1$ to n, that is, $D^T b > 0$, or b must be in the interior of the polar cone of Pos (D), Pos ($(D^T)^{-1}$).

Geometrically, Corollary 1 says that every CP-ray for Pos (D) must make a strict acute angle with each of the generator rays of Pos (D).

COLLARY 2: Let D be a real nonsingular square matrix of order n. The set of CP-points for Pos (D) is a subset of the intersection of the interiors of Pos (D) and Pos ($(D^T)^{-1}$).

PROOF: Follows from the definition and Corollary 1. □

Let D be a real nonsingular square matrix of order n. When $n = 2$, in Section 3 we have seen that the set of CP-points of Pos (D) is actually (interior of Pos (D)) ∩ (interior of Pos ($(D^T)^{-1}$)). In Sections 7, 8 we establish that this result also holds for some special classes of matrices D when $n > 2$. Also, when $n = 2$, the set of CP-points of Pos (D) is either the interior of Pos (D) (when Pos (D) is an acute or right angle), or the interior of Pos ($(D^T)^{-1}$) (when Pos (D) is an obtuse angle). In Sections 7, 8 we derive some necessary and sufficient conditions on D, for these properties to hold, when $n > 2$.

THEOREM 2: Let M be a PD symmetric matrix of order n, and let D be a square matrix satisfying $D^T D = M$. There exists a PC-vector for M if Pos (D) is a CP-owner.

PROOF: Follows directly from Theorem 1.

$\square$

THEOREM 3: Let M be a P-matrix of order n. The set of all PC-vectors for M is either $\emptyset$ or is an open polyhedral cone.

PROOF: A vector $p \in R^n$ is a PC-vector for M iff it is a feasible solution of the system

$$\begin{aligned} p &> 0 \\ (M_{JJ})^{-1} p_J &> 0, \text{ for all } \emptyset \neq J \subset \Gamma \end{aligned} \tag{14}$$

(14) is a finite system of strict linear inequalities, and its set of feasible solutions is the set of PC-vectors for M. Hence, if this set is nonempty, it is an open polyhedral cone.

$\square$

THEOREM 4: Let D be a nonsingular square matrix of order n. The set of CP-points for Pos (D) is either empty, or is an open polyhedral cone.

PROOF: Let $M = D^T D$, and let K be the set of PC-vectors for M. By Theorem 1, the set of all CP-points for Pos (D) is $\{x: x = (D^T)^{-1} p, p \in K\}$ and this is an open polyhedral cone when it is nonempty, since the same property holds for K. $\square$

5. PC-VECTORS FOR P-MATRICES OF ORDER 2.

Suppose $M = (M_{ij})$ is a P-matrix of order 2. Then from the definitions, $p = (p_1, p_2)^T$ is a PC-vector for M iff it is feasible to (15).

$$
\begin{aligned}
m_{22}p_1 - m_{12}\;\; p_2 &> 0\\
-m_{21}p_1 + m_{11}\;\; p_2 &> 0\\
p_1 &> 0\\
p_2 &> 0
\end{aligned}
\tag{15}
$$

Using Gordan's theorem of the alternatives [7, 11], it can be verified that (15) is always feasible. Hence, every P-matrix M of order 2 has a PC-vector p, which can be found by solving (15).

6. THE HEREDITARY FEATURE OF THESE PROPERTIES.

Let D be a nonsingular square matrix of order n.

THEOREM 5: If Pos (D) is a CP-owner, so is every face of Pos (D).

PROOF: Consider the facet Pos $(D_{.2}, \cdots, D_{.n}) = K_1$ of Pos (D) and the facetal hyperplane H_1 containing it. Suppose b is a CP-point for Pos (D). Let b^1 be the orthogonal projection of b in H_1. Since b is a CP-point for Pos (D), b^1 is in the relative interior of K_1. By Pythogoras theorem, $\| x - b \|^2 = \| x - b^1 \|^2 + \| b - b^1 \|^2$ for all $x \in H_1$. So, if F is any face of K_1, the orthogonal projections of b, b^1 in the linear hull of F, are the same. So, by the hypothesis, b^1 is a CP-point for K_1. A similar proof holds for all facets of Pos (D), so all facets of Pos (D) are CP-owners, if Pos (D) is a CP-owner. Repeating this argument, we conclude that all faces of Pos (D) are CP-owners if Pos (D) is.

THEOREM 6: If every interior point of Pos (D) is a CP-point for it, then for every face of Pos (D), every relative interior point is a CP-point.

PROOF: Consider the facet $K_1 =$ Pos $(D_{.2}, \cdots, D_{.n})$ of Pos (D). Let b^1 be any point in the relative interior of K_1. Erect the inward normal at b^1 to the facetal hyperplane containing K_1 and let b be a point on this normal in the interior of Pos (D). By the hypothesis of the theorem, b is a CP-point for Pos (D), so by the arguments in the proof of Theorem 5, b^1 is a CP-point for K_1. So, every point in the relative interior of K_1 is a CP-point for K_1. A Similar proof holds for all facets of Pos (D); so, every facet inherits the property that all points in its relative interior are its CP-points. Repeating this argument, we conclude that every relative interior point of any face of Pos (D) is a CP-point of that face, if every interior point of Pos (D) is its CP-point. □

THEOREM 7: Let M be a P-matrix of order n. If M has a PC-vector, so does every principal submatrix of M.

PROOF: Let p be a PC-vector for M. Then, for every $J \subset \Gamma$, it follows directly from the definition; that p_J is a PC-vector for the principal submatrix M_{JJ} of M corresponding to the subset J. □

7. CP-POINTS FOR POS (D) WHEN EVERY PAIR OF GENERATORS MAKE AN OBTUSE ANGLE.

THEOREM 8: Let D be a real nonsingular square submatrix of order n. If every pair of generators of Pos (D) make a non-acute (i.e., either obtuse or right) angle, then Pos (D) is a CP-owner, and the set of CP-points of Pos (D) is the interior of the polar cone of Pos (D), Pos $((D^T)^{-1})$.

PROOF: Suppose every pair of generators of Pos (D) makes a non-acute angle. So $(D_{.i})^T D_{.j} \leqq 0$ for all $i \neq j$. Therefore, $M = D^T D$ is a P-matrix which is also a Z-matrix, and hence an M-matrix. By the results in [3] this implies that $M^{-1} \geqq 0$, and by the same argument $(M_{JJ})^{-1} \geqq 0$ for all $\emptyset \neq J \subset \Gamma$. Let $p \in R^n, p > 0$. Let $M^{-1} p = y = (y_i)$. Since $M^{-1} \geqq 0, p > 0$, we have $y \geqq 0$. We will show that $y > 0$. Suppose not. Assume $y_i = 0$ for some i. We have $My = p > 0$. So, $p_i = M_{i.}\, y > 0$. But $M_{i.}\, y = \sum_{(j \neq i)} m_{ij}\, y_j$ (since $y_i = 0$ by assumption) $\leqq 0$ (since $y_j \geqq 0$ for all j and $m_{ij} \leqq 0$ for all $j \neq i$), a contradiction. So, we must have $y = M^{-1} p > 0$ for all $p > 0$. A similar argument shows that $(M_{JJ})^{-1} p_J > 0$ for all $\emptyset \neq J \subset \Gamma$, $p > 0$. Hence every $p \in R^n$, $p > 0$, is a PC-vector for $M = D^T D$. Consequently by Theorem 1, all b in the interior of Pos $((D^T)^{-1})$ are CP-points for Pos (D). By Corollary 1, this implies that the set of all CP-points for Pos (D) in this case is the interior of Pos $((D^T)^{-1})$. □

An alternate proof of this Theorem based on spherical geometric arguments is given in Section 10.

SETS OF CP-POINTS FOR POS (D) AND ITS POLAR CONE ARE THE SAME WHEN EVERY PAIR OF GENERATORS OF POS (D) MAKE AN OBTUSE ANGLE.

Let D be a real nonsingular square matrix of order n. D^TD is a Z-matrix iff every pair of generators of Pos (D) make an obtuse angle. Suppose these properties hold. By the results in [3] this implies that $(D^TD)^{-1} \geqq 0$. Hence, as discussed in Section 3, in this case the polar cone of Pos (D), Pos $((D^T)^{-1}) \subset$ Pos (D). Also, applying the definitions, it can be verified that all the dihedral angles associated with the polar cone Pos $((D^T)^{-1})$ are acute, and hence by the results of the following Section (Section 8), the set of CP-points for the Polar Cone Pos $((D^T)^{-1})$ is its interior. Hence, in this case, the sets of CP-points for both Pos (D) and its polar cone are the same, they are the set of interior points of the Polar cone, Pos $((D^T)^{-1})$. Conversely, we show in Section 9 (Theorem 17) that if every interior point of the polar cone is a CP-point for it, then each of those points is also a CP-point for Pos (D) and D^TD must be a Z-matrix.

8. CP-POINTS FOR POS (D) WHEN EVERY PAIR OF GENERATORS MAKE AN ACUTE ANGLE.

In this section, D is a real nonsinular square matrix of order n.

In Section 3, we have seen that if $n = 2$, and pos (D) is an acute or right angle (i.e., $(D_{.1})^T D_{.2} \geqq 0$), every interior point of Pos (D) is a CP-point for it. An intuitive generalization of this result to higher dimensions states that if the rays of $D_{.i}$ and $D_{.j}$ make an acute or right angle (i.e., $(D_{.i})^T D_{.j} \geqq 0$) for all $i \neq j$, then Pos (D) is a CP-owner. This result is true for $n = 2$, and established for $n = 3$ in Section 10 using spherical geometry. However, this result may be false for $n \geqq 4$. Using the arguments in Section 10, or using the result in Theorem 1, it can be verified that for the following matrix D

$$D = \begin{pmatrix} -1 & 1 & 20 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 5 & 5 & 5 \end{pmatrix} \tag{16}$$

even though $(D_{.i})^T D_{.j} > 0$ for all $i \neq j$, Pos (D) is a CP-lacker.

In order to guarantee that Pos (D) is a CP-owner in this case, we need to impose more conditions on the matrix D. Our work on this class of matrices D was motivated by the following conjecture made by Soo Y. Chang.

CONJECTURE: Let $K_j = \text{Pos}(D_{.1}, \cdots, D_{.j-1}\ D_{.j+1}, \cdots, D_{.n})$ and H_j the facetal hyperplane of Pos (D) containing K_j. If for each $j = 1$ to n, the orthogonal projection of the generator Pos $(D_{.j})$ in H_j lies in the relative interior of K_j, then Pos (D) is a CP-owner.

While it may not be immediately apparent, we will show that the hypothesis in the conjecture implies that every pair of generators of D make an acute angle, so this conjecture legitimately belongs in this section. We will provide a proof of a stronger form of this conjecture and derive several related results.

For ease of reading, we will summarize the notation for this section. K_j and H_j are as stated in the above conjecture. $\beta = (\beta_{ij}) = D^{-1}$. Then,

$$H_j = \{ x:\ \beta_{j.}\, x = 0 \} \tag{17}$$

$$\text{Pos}(D) = \{ x:\ \beta_{j.}\, x \geqq 0, \quad j = 1 \text{ to } n \} \tag{18}$$

$$K_j = \{ x:\ \beta_{i.}\, x \geqq 0, \quad i = 1 \text{ to } n,\ i \neq j;\ \text{and } \beta_{j.} x = 0 \} \tag{19}$$

$G_{.j} = (\beta_{j.})^T / \| \beta_{j.} \|$,

a nonzero point on the inward normal to H_j (20)

Consider the hyperplane $\{ x:\ a_1 x_1 + \cdots + a_n x_n = a x = d \}$ and let $\bar{x} \in R^n$. The orthogonal projection of $\bar{x}$ in this hyperplane is $\bar{x} + a^T \left(\frac{d - a\bar{x}}{\|a\|^2} \right)$.

We will now investigate the simplicial cones Pos (D) satisfying one of these properties.

5. For each $j = 1$ to n, the orthogonal projection of the ray Pos $(D_{.j})$ in H_j is in the relative interior of K_j

6. For each $j = 1$ to n, the orthogonal projection of the ray Pos $(D_{.j})$ in H_j is in K_j.

7. All the dihedral angles associated with Pos(D) are acute.

8. All the dihedral angles associated with Pos (D) are non-obtuse (acute or right).

9. Every interior point of Pos (D) is a CP-point for it.

THEOREM 9: If property 5 holds, all the $\binom{n}{2}$ dihedral angles associated with Pos (D) are acute, and conversely.

PROOF: The orthogonal projection of $D_{.1}$ on H_1 is $D_{.1} - (\beta_{1.})^T (\beta_{1.} D_{.1}) / \|\beta_{1.}\|^2, = D_{.1} - G_{.1}$ (since $\beta_{1.} D_{.1} = 1$, as $\beta = D^{-1}$). Under property 5 the orthogonal projection of $D_{.1}$ on H_1 is in the relative interior of K_1, and hence from (19), $\beta_{i.} (D_{.1} - G_{.1}) > 0$ for all $i \neq 1$. But $\beta_{i.} D_{.1} = 0$ for all $i \neq 1$, since $\beta = D^{-1}$. So $\beta_{i.} G_{.1} < 0$ for all $i \neq 1$, that is, $(G_{.i})^T G_{.1} < 0$, for all $i \neq 1$. In the same way, under property 5, we have

$$(G_{.i})^T G_{.j} < 0 \text{ for all } i \neq j$$

that is, all the dihedral angles associated with Pos (D) are acute.

The converse is established by essentially reversing the steps of the proof. □

THEOREM 10: If property 6 holds, all the $\binom{n}{2}$ dihedral angles associated with Pos (D) are non-obtuse (i.e., acute or right), and conversely.

PROOF: Similar to the proof of Theorem 9. □

Thus, properties 5 and 7 are equivalent. Likewise, properties 6 and 8 are equivalent.

LEMMA 4: Let $\bar{x}$ be any interior point of Pos (D). If the orthogonal projection of $D_{.1}$ in H_1 is in the relative interior of K_1, the orthogonal projection of $\bar{x}$ in H_1 is also in the relative interior of K_1.

PROOF: Suppose the orthogonal projection of $D_{.1}$ in H_1 is in the relative interior of K_1. Then, from the proof of Theorem 9, we have, $\beta_{i.}(\beta_{1.})^T < 0$ for all $i \neq 1$.

Now, the orthogonal projection of $\bar{x}$ in H_1 is $\hat{x} = \bar{x} - (\beta_{1.})^T (\beta_{1.}\bar{x}) / \|\beta_{1.}\|^2$. We have, for $i \neq 1$,

$$\beta_{i.}\hat{x} = \beta_{i.}\bar{x} + (-\beta_{i.}(\beta_{1.})^T)(\beta_{1.}\bar{x}) / \|\beta_{1.}\|^2$$

$$> 0$$

because $\beta_{i.}\bar{x} > 0$ (from (18), since $\bar{x}$ is in the interior of Pos (D)) for all i, and $(-\beta_{i.}(\beta_{1.})^T) > 0$ (established above). So, $\hat{x}$ is in the relative interior of K_1, completing the proof of this lemma. □.

THEOREM 11: If property 6 holds, then every interior point of Pos (D) is a CP-point for it, and conversely.

PROOF: The proof is essentially similar to the proof of Lemma 4. Suppose property 6 holds. Let $\bar{x}$ be an interior point of Pos (D). Let F be a proper face of Pos (D). So, from (18), there must exist a $\emptyset \neq J \subset \Gamma$ such that

$$F = \{x : \beta_{J.}x = 0, \; \beta_{i.}x \geqq 0, i \in \Gamma \setminus J\}. \tag{21}$$

Let $|J| = r$. So $\beta_{J.}$ is of order $r \times n$ and of full row rank. Hence the matrix

$A = \beta_{J.} (\beta_{J.})^T$ is PD symmetric. By Theorem 10, we have $\beta_{i.} (\beta_{.j})^T \leqq 0$, for all $i \neq j$. This implies that A is a Z-matrix, and hence an M-matrix. By the results in [3], this implies that $A^{-1} \geqq 0$.

The orthogonal projection of $\bar{x}$ in the linear hull of F is the optimum solution of the problem

$$\begin{aligned} &\text{minimize } (x - \bar{x})^T (x - \bar{x}) \\ &\text{subject to } \beta_{J.}\ x = 0. \end{aligned}$$

which is $\hat{x} = \bar{x} - (\beta_{J.})^T A^{-1} \beta_{J.} \bar{x}$.

Let $i \in \Gamma \setminus J$. We have

$$\begin{aligned} \beta_{i.} \hat{x} &= \beta_{i.} \bar{x} - \beta_{i.} (\beta_{J.})^T A^{-1} \beta_{J.} \bar{x} \\ &> 0 \end{aligned}$$

because $\beta_{i.} \bar{x} > 0$ (since $\bar{x}$ is in the interior of Pos (D)), $\beta_{i.} (\beta_{J.})^T \leqq 0$ (since $\beta_{i.} (\beta_{j.})^T \leqq 0$ for all $j \neq i$), $A^{-1} \geqq 0$ (established above), and $\beta_{J.}\ x > 0$ (since x is in the interior of Pos (D)). So, $\hat{x}$ is in the relative interior of F, that is, F is a projection face relative to x. Since this holds for all faces F of Pos (D), $\bar{x}$ is a CP-point for Pos (D). Hence every interior point of Pos (D) is a CP-point for pos (D) in this case.

To prove the converse, suppose every interior point of Pos (D) is a CP-point for it. Suppose there is a j such that the orthogonal projection of $D_{.j}$ in H_j is not in K_j, say for $j = 1$. Since orthogonal projection is a continuous operation, we can find an open ball B containing $D_{.1}$ such that the orthogonal projection of every point inside B in H_1 is outside of K_1. Since B is an open ball containing $D_{.1}$, it contains some points from the interior of Pos (D), and the orthogonal projection of these points are outside K_1, contradicting the hypothesis. So, if every interior point of Pos (D) is a CP-point, property 6 must hold. □

Hence, properties 6 and 9 are equivalent.

We can think of each face of Pos (D) as being a full dimensional simplicial cone in its linear hull. If F is an r-dimensional face of Pos (D), we can define the $\binom{r}{2}$ dihedral angles of F relative to its linear hull, just as we defined the dihedral angles for Pos (D).

We will now show that properties 6 and 8 are facially hereditary, in the sense that if Pos (D) has the property, then every face of Pos (D) also has the corresponding property.

THEOREM 12: If the simplicial cone Pos (D) has property 6, then every face of Pos (D) has the corresponding property.

PROOF: This follows from Theorems 11 and 6. □

THEOREM 13: If all the dihedral angles defined by pairs of facets of the simplicial cone Pos (D) are non-obtuse, every face of Pos (D) also has the same property.

PROOF: Follows directly from Theorems 11, 10 and 12. □

THEOREM 14: For any simplicial cone Pos (D), properties 6, 8 and 9 are equivalent, and these properties are inherited by all faces of Pos (D).

PROOF: Follows from Theorems 10, 11, 12 and 13. □

We will now explain the relationship of these properties to the condition mentioned in the heading of this section, and the role that Z-matrices play in these properties. From the results in this section, we see that the basic condition for properties 6 or 8 or 9 to hold is that

$$\beta_{i.} (\beta_{j.})^T \leqq 0 \text{ for all } i \neq j. \qquad (22)$$

that is, that $\beta\beta^T$ is a Z-matrix. Since β is nonsingular, this is equivalent to requiring that $\beta\beta^T$ be an M-matrix. But $\beta\beta^T = (D^{-1})(D^{-1})^T = (D^TD)^{-1} = M^{-1}$ where $M = D^TD$. So, if (22) holds, $(D^TD)^{-1}$ is an M-matrix, and by the results in [3], $D^TD \geqq 0$, in other words, every pair of generators for Pos (D) make an acute (or non-obtuse, to be precise) angle. Hence all the results in this section relate to a subclass of simplicial cones in which every pair of generators make an acute angle.

One noteworthy feature. Let $D^TD = M$. In Section 7 we dealt with the case where M is a Z-matrix. In this Section 8, we are dealing with the case where M^{-1} is a Z-matrix. We now provide a theorem which summarizes this section.

THEOREM 15: Every interior point of Pos (D) is a CP-point for it, iff $(D^TD)^{-1}$ is a Z-matrix. Also, properties 6, 8, and 9, and the condition that $(D^TD)^{-1}$ is a Z-matrix, are all equivalent.

PROOF: Requiring $(D^TD)^{-1}$ to be a Z-matrix is equivalent to requiring property 8 as explained above. So, these results follow from Theorem 14. □

The simplicial cone Pos (D) is called a regular simplicial cone if the convex hull of $\{ \frac{D_{.j}}{\|D_{.j}\|} : j = 1 \text{ to } n \}$ is a regular simplex. Property 9 of course holds for regular simplicial cones, and this is of principal interest when referring to regularity in simplicial cones. So, the conditions given in Theorem 15 (namely that $(D^TD)^{-1}$ should be a Z-matrix) provide a proper generalization of the traditional concept of regularity in simplicial cones. It identifies the class of simplicial cones satisfying property 9.

9. SIMPLICIAL CONES POS (D) WITH ALL DIHEDRAL ANGLES OBTUSE.

Let D be a real nonsingular square matrix of order n.

THEOREM 16: If n = 3, and all the dihedral angles associated with Pos (D) are obtuse, then the circumcenter ray of Pos (D) is a CP-ray for it.

PROOF: From the definition of the dihedral angles associated with Pos (D) (Section 3), we know that they are all obtuse iff

$$(D^TD)^{-1} \geqq 0 \qquad (23)$$

Let $\tau = (\tau_1, \cdots, \tau_n)^T$ where $\tau_j = \| D_{.j} \|$, j = 1 to n, and

$$b = (D^T)^{-1} \tau \tag{24}$$

b is the circumcenter for Pos (D). Since $D^{-1} b = (D^T D)^{-1} \tau > 0$ (because $\tau > 0$, (23), and since $(D^T D)^{-1}$ is a PD symmetric matrix, its main diagonal is >0), b is in the interior of Pos (D).

Let $M = (m_{ij}) = D^T D$. We will now show that

$$(M_{JJ})^{-1} \tau_J > 0 \text{ for all } J \subset \{1, 2, 3\}. \tag{25}$$

Since $M^{-1} \tau = D^{-1} b > 0$, we know that (25) holds for $J = \{1, 2, 3\}$. When $J = \{1, 2\}$, it can be verified that $(M_{JJ})^{-1} \tau_J = (\tau_2, \tau_1)^T / (\tau_1 \tau_2 + (D_{.1})^T (D_{.2}) > 0$ because τ_1 and τ_2 are >0 and $\tau_1 \tau_2 + (D_{.1})^T D_{.2} > 0$ by Cauchy-Schwartz inequality, so (25) holds. By symmetry, (25) holds whenever $|J| = 2$. When $J = \{j\}$, $(M_{JJ})^{-1} \tau_J = (1/\tau_j) > 0$, for all j = 1, 2, 3, so (25) holds. Hence τ is a PC-vector for M, and by Theorem 1, the circumcenter $b = (D^T)^{-1} \tau$ is a CP-point for Pos (D). So, the circumcenter ray is a CP-ray for Pos (D) in this case.

□

For any n, if all the dihedral angles associated with Pos (D) are obtuse, (23) holds. Under (23), the circumcenter ray for pos (D) is an interior ray for Pos (D). Unfortunately, under these conditions, the circumcenter ray is not guaranteed to be a CP-ray, if $n \geqq 4$. See Section 10.

Thus, when $n \geqq 4$, to guarantee that a CP-point exists for Pos (D), conditions (23) are not adequate, we need to impose more conditions. We have the following theorem.

THEOREM 17: If every face of Pos (D) satisfies the property that all the dihedral angles associated with it are obtuse, then the set of CP-points for Pos (D) is the interior of Pos ($(D^T)^{-1}$), and conversely.

PROOF: Let $M = D^T D$. Suppose, for every face of Pos (D) the dihedral angles associated with it are obtuse. This implies that $(M_{JJ})^{-1} \geqq 0$ for all $J \subset \Gamma$. Also, since $(M_{JJ})^{-1}$ is a PD-symmetric matrix, its main diagonal is >0. Hence every $p \in R^n$,

$p>0$ is a PC-vector for M. By Theorem 1, every interior point of Pos $((D^T)^{-1})$ is a CP-point for Pos (D). By Corollary 1, we conclude that the set of CP-points for Pos (D) in this case is the interior of Pos $((D^T)^{-1})$.

To prove the converse, suppose every interior point of Pos $((D^T)^{-1})$ is a CP-point for Pos (D). By Theorem 1, every vector $p \in R^n$, $p>0$ is a PC-vector for $M = D^T D$. Hence for every $J \subset \Gamma$, $(M_{JJ})^{-1} p_J > 0$ for all $p_J > 0$. This implies that $(M_{JJ})^{-1} > 0$ for all $J \subset \Gamma$, that is, that the dihedral angles associated with the face of Pos (D) corresponding to $J \subset \Gamma$ are all obtuse. Since this holds for all $J \subset \Gamma$, the dihedral angles associated with every face of Pos (D) are obtuse in this case. □

It can be verified that the result in Theorem 8 is a special case of the general result in Theorem 17.

We now have the following theorem relating to the results obtained in Section 7.

THEOREM 18: Let D be a real nonsingular square matrix of order n. Every interior point of the polar cone of Pos (D) is a CP-point for the polar cone iff every pair of generators of Pos (D) make an obtuse angle, that is, iff $D^T D$ is a Z-matrix.

PROOF: If $D^T D$ is a Z-matrix, the results in Section 7 imply that every interior point of the polar cone is a CP-point for it. Conversely, if every interior point of the polar cone, Pos $((D^T)^{-1})$, is a CP-point for it, by applying Theorem 15 to the polar cone, we conclude that $(((D^T)^{-1})^T (D^T)^{-1})^{-1} = D^T D$ must be a Z-matrix. □

10. THE SPHERICAL VIEW

The conical interpretation of the LCP has been eminently successful in producing significant advances in the theory. Recently the work of L. M. Kelly and L. T. Watson [5, 6] has demonstrated the advantages, both heuristic and substantive, of a further refinement of this conical analysis to what might be called the spherical view. A number of investigators [2] subsequently employed this technique with profit.

The spherical view is easily described once the conical is understood. Many questions concerning simplicial cones can be resolved by considering corresponding questions concerning the "traces" of these cones on the unit sphere centered at the common vertex (origin) of the cones. The trace (or spherical section) of a simplicial cone is, of course, a spherical simplex. Thus in the analysis of 3×3 matrices we are often led to an analysis of triangular cones in R^3 and this in turn might be conveniently viewed as a problem concerning spherical triangles in S^2. The 2-sphere in R^3, the boundary of the unit ball, endowed with the geodesic (shorter arc) metric is denoted S^2, and in general the k-sphere in R^{K+1} with this metric is S^k.

There has been an elaborate study of spherical geometry in its own right, that is, without regarding S^k as embedded in R^{K+1}, and we will take advantage of some of this elementary lore without apology.

The spherical view has the advantage in some instances of reducing the heuristics from 3 dimensions to 2 or as in [5, 6] from 4 to 3. It has the further advantage of making more directly available the classical pole-polar lore much of which is framed in the spherical setting.

We now illustrate some of these claims in the context of the previous sections. This section will be strictly geometric allowing for a simplification of some notation. The rays of a simplicial cone in R^n with vertex at the origin will be denoted r_i. If rays $r_1, r_2, \cdots, r_k$ in R^n are affinely independent then the convex hull of the rays is a simplicial cone of dimension k. The intersection of such a cone with the unit sphere S in R^n is a spherical simplex of dimension $k - 1$. As noted above S endowed with the shorter arc metric will be denoted S^{n-1}. Simplicial cones will generally be denoted C and the associated spherical simplex is V. If r_i is a generator ray of C, $v_i = r_i \cap S$ is a vertex of V. C^* denotes the polar cone of C and V^* is the spherical simplex associated with C^*. V^* is known as the polar simplex of V.

While we could define a CP-point of V as the intersection with S of a CP-ray of C we prefer to describe it as a point interior to V which projects (spherically) "internally" on all the faces of V. We hasten to remark that in the case of zero faces (i.e., vertices) "P projects internally on v_i" is taken to mean that the spherical distance $\overline{Pv_i} < \pi/2$. The spherical distance between points P and Q of S^n will be denoted $\overline{PQ}$.

In the familiar setting of S^2, V can have at most 3 vertices in which case it is a spherical triangle and the associated C in R^3 is a triangular cone. The edges of V have the same radian measures as the corresponding generator angles of C while the interior angles of the triangle have the same measures as the dihedral angles of C. More generally the dihedral angles of C have the same measure as the corresponding dihedral angles of the simplex V.

EXAMPLE 1. For a spherical triangle V on S^2 with acute edges the center of the inscribed circle (the incenter) is a CP-point. Certainly every triangle has an incenter

b and the spherical distance $\overline{bv_i} < \pi/2$ if the edges are acute. See Figure 10.

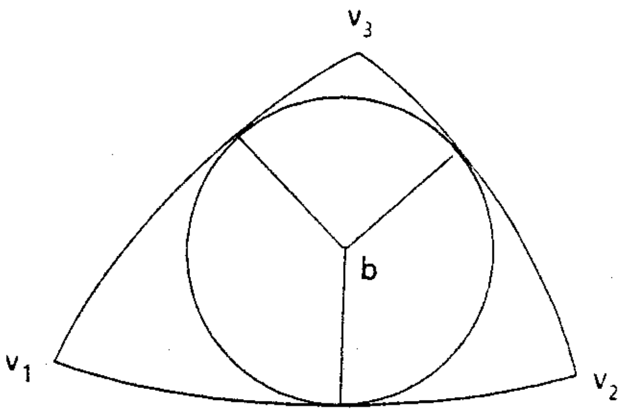

Figure 10

EXAMPLE 2. Fig. 11 shows a triangle V on S^2 with no CP-point.

Consider V with $\overline{v_1 v_2} = \pi - \varepsilon$, $\overline{v_1 v_3} = \pi - 2\varepsilon$, $\overline{v_2 v_3} = 3\varepsilon$ for very small $\varepsilon > 0$. Any CP-point for V must be interior to $\Delta\, v_2 v_3 D$ where $< v_2 v_3 D$ is a right angle and similarly it must be in the right triangle $v_1 v_3 E$, where $< v_1 v_3 E$ is right. Thus any CP-point of V must be interior to the triangle $v_3 DE$. But for very small $\varepsilon > 0$ points in this triangle are clearly much farther than $\pi/2$ from v_1. This example easily leads to a 3×3 matrix similar to the one in (3) in Section 3.

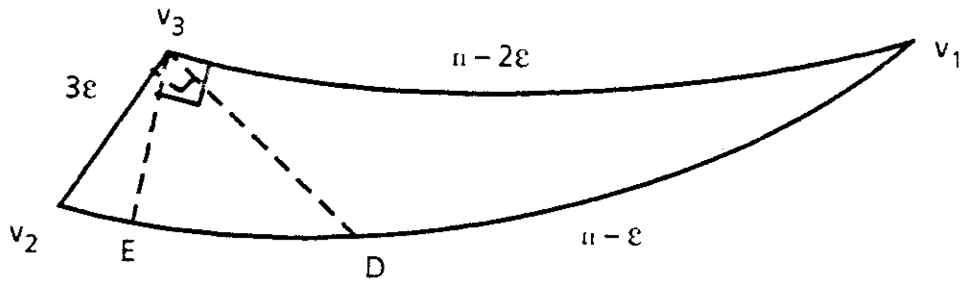

Figure 11

EXAMPLE 3. Does Example 1 generalize? That is, is it true that for a simplex V with acute edges, the incenter is always a CP-point? The matrix in (16) in Section 8 shows that this is not true for simplices in S^3, i.e., for spherical tetrahedra. We show here how to discover and construct such examples.

A "very small" spherical simplex is "very nearly" euclidean so it should be clear that the production of a euclidean simplex with no CP-point will lead to the production of a spherical simplex with no CP-point. To produce such a euclidean simplex in R^3 we start with a degenerate tetrahedron in R^2, specifically a planar parallelogram ABCU in a horizontal plane L with angles BAC and ACU both, say, 100°. Now elevate U slightly above L, rotating ΔACU about line AC. We claim the resulting tetrahedron has no CP-point.

For, any such point would have to be in the right triangular prism with base ΔABE where $< EAB = \pi/2$. It must also be very close to L.

Similar reasoning shows that it must be in the right triangular prism with base ΔFCU. The intersection of these two prisms is far above L if U is sufficiently close to L. This shows that there must be an analagous small spherical tetrahedron in S^3 and thus a simplicial cone in R^4 with small generator angles and no CP ray.

On the other hand we have already shown (Section 7) that if all generator angles of a simplicial cone are "large" i.e., obtuse then the cone has a CP ray. We will sketch a simple spherical argument showing this in Theorem 21 later on.

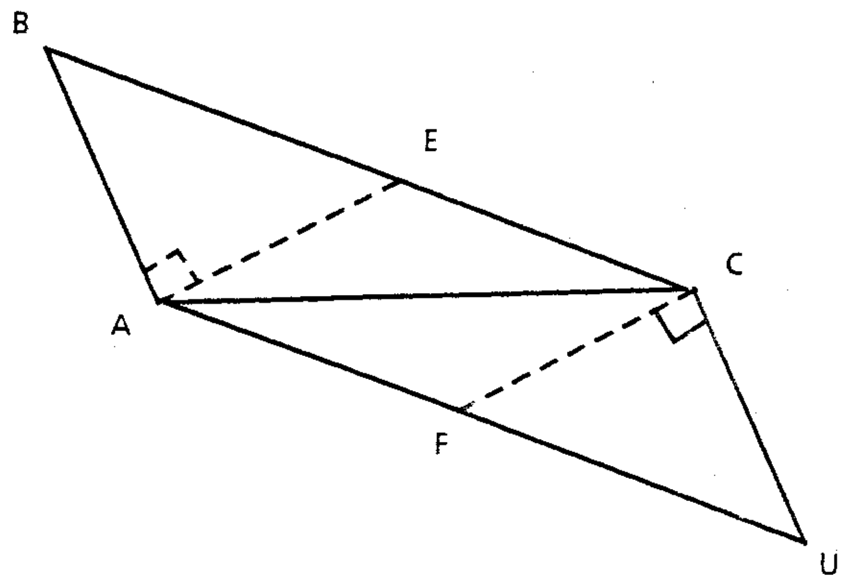

Figure 12

Meanwhile we turn our attention to triangular cones with large (obtuse) dihedral angles in R^3 which means that we will examine spherical triangles V, on S^2, with obtuse angles. We claim that such triangles must have a CP point and in fact the center of the circumscribed circle is one such point. Let $V = \Delta v_1\, v_2\, v_3$ in Figure 13 be one such triangle.

Consider the triangle with vertices v_1, v_2, v_3^* where $< v_3^*\, v_1\, v_2 = < v_3^*\, v_2\, v_1 = \pi/2$, as in Figure 13. v_3^* is clearly interior to V.

Now the circumradius of V is less than $\pi/2$ in length and the locus of points P with

$\overline{Pv_1} = \overline{Pv_2} < \pi/2$ is the segment $\overline{V_3^*m}$. Hence the circumcenter, T, of V is interior to V. Now T is easily seen to project into the midpoints of the edges, so T is a CP point of V.

We might be tempted to hope that obtuse dihedrals would always produce a nonempty set of CP points but we now show that this projected theorem is false even in R^4. Recall that the projection of a CP point of V onto a facet F of V is a CP point of F and that the triangle in Example 2 has no CP point. If we can show that this triangle can be made a facet of a spherical tetrahedron with all 6 of its dihedrals

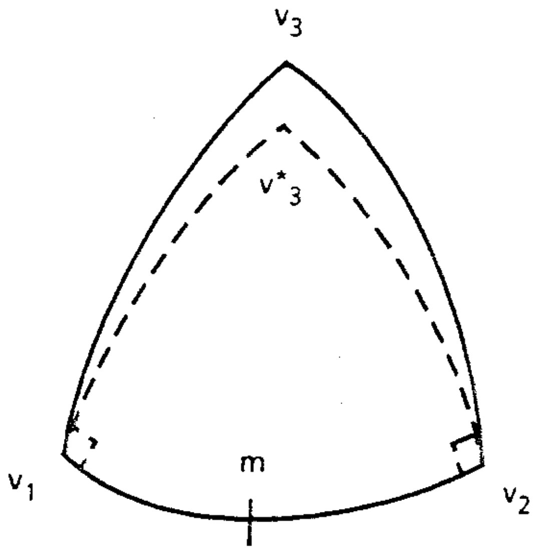

Figure 13

obtuse, then we will have an example of a simplex V in S^3 with all obtuse dihedrals and no CP point.

Such a tetrahedron is easy to construct as follows. We start with a degenerate tetrahedron on the equator of S^3 one of whose facets is the triangle, Δ, of Example 2 - and the fourth vertex is a point E on the equator diametrically opposite to an interior point of Δ. The four facets of this degenerate tetrahedron cover the equator and have disjoint relative interiors. Their (degenerate) dihedrals are all 180°. Now elevate E slightly to E in the northern hemisphere of S^3. The resulting nondegenerate tetrahedron has obtuse dihedrals and no CP point. Using this technique, we get the tetrahedron which is the trace of the simplicial cone Pos (D), where D is

$$D = \begin{pmatrix} -1 & 1 & 20 & -20 \\ 0 & 3 & 1 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1/3 \end{pmatrix}$$

and it can be verified that all dihedral angles associated with Pos (D) are obtuse, and yet Pos (D) has no CP-point.

ALTERNATE SPHERICAL PROOFS OF A FEW THEOREMS

A necessary and, in some interesting cases, sufficient condition that a point be a CP point for V is that it be interior to V^*. Thus we should have a good grasp of the geometry of the polar cone. We list a number of properties well known in the literature or easily verified, which will be useful. For each i, exactly one vertex of V^* is closer to v_i than $\pi/2$. We label this vertex v_i^*; $\overline{v_i^* v_j} = \pi/2$ for $i \neq j$. This sets up a natural 1 – 1 correspondence between the vertices of V and V^*.

Each edge of a simplex intersects the interior of exactly one dihedral angle of the simplex. The edge is said to be opposite the dihedral angle.

<u>Classical result 1</u>. If α is the measure of $\overline{v_i v_j}$ and β is the measure of the dihedral angle of V^* opposite $\overline{v_i^* v_j^*}$, then $\alpha + \beta = \pi$.

<u>Classical result 2</u>. $(V^*)^* = V$.

<u>Classical result 3</u>. The orthogonal projection of V^* on any face F of V is F^*, the polar simplex of F.

<u>Classical result 4</u>. $x \in V$, $y \in V^*$ implies $\overline{xy} \leq \pi/2$.

<u>Notation</u>. k affinely independent points in S^n determine an S^{k-1} which we call the spherical span of the k points. If A is such a set we denote its spherical span by $\hat{A}$. More generally any set in S^n determines a spherical span which we denote similarly.

THEOREM 19: $V \cap V^* = \emptyset$.

PROOF: Consider the function $f(x) = \Sigma\ \overline{v_i x}$, $x \in V^*$. This function assumes its minimum over V^* at some point P of V^*. We claim P must be in V. If not, P must be on the opposite side of $\hat{F}_i$ from v_i for some vertex v_i of V. See Figure 14.

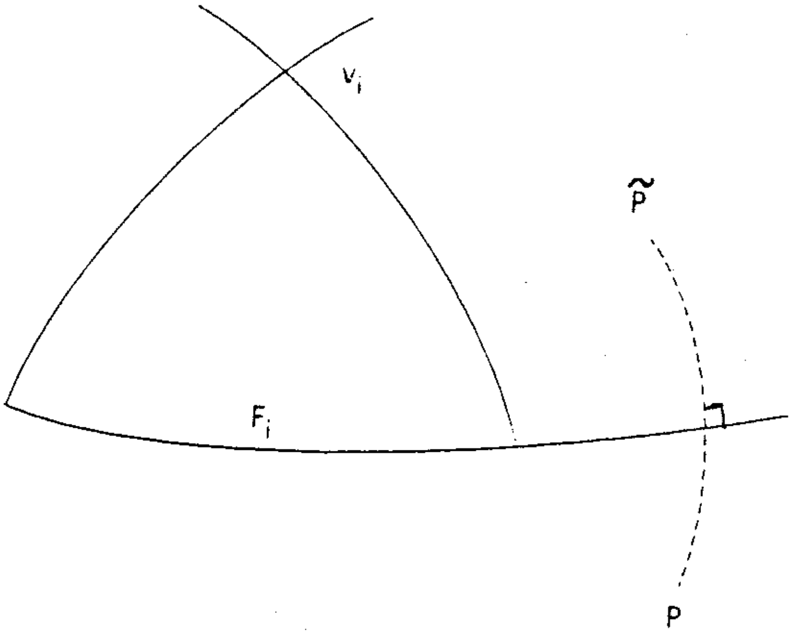

Figure 14

Let $\tilde{P}$ be the reflection of P in $\hat{F}_i$. $\overline{v_i\tilde{P}} < \overline{v_iP}$, while $\overline{v_j\tilde{P}} = \overline{v_jP}$, $j \neq i$. Hence $f(\tilde{P}) < f(P)$, a contradiction. □

This is probably a classical result as well, but we could not locate it in the literature. Note this argument does not show that V and V^* have interior points in common but this was shown in Section 3.

THEOREM 20: If all edges of V are obtuse, $V^* \subset \text{int } V$.

PROOF: First note the following simple but useful fact. If all edges with end point v_i are obtuse and x is any point in F_i then $\overline{v_ix}$ is obtuse. For, if we envision v_i as the north pole, then all v_j, $j \neq i$ are in the southern hemisphere and hence so is F_i. Thus

$\overline{v_i x} > \pi/2$.

A similar argument shows that if all edges with end point v_i are acute then $v_i x < \pi/2$, for $x \in F_i$.

Now to the theorem. Induce on the dimension. Let $P \in V^*$. If $P \in \hat{F}_i$ then by induction it must be interior to F_i. But this is impossible by our opening observation. Hence no element of V^* is in any $\hat{F}_i$. Suppose now that P is on the opposite side of F_i from v_i. Let $\tilde{P}$ be the reflection of P in $\hat{F}_i$. See Figure 15. As in Theorem 19, we see that $\tilde{P} \in V^*$. But by convexity, the midpoint Q of $\overline{P\tilde{P}}$ is in V^*. But it is also in $\hat{F}_i$ which is impossible. Thus P must be interior to V.

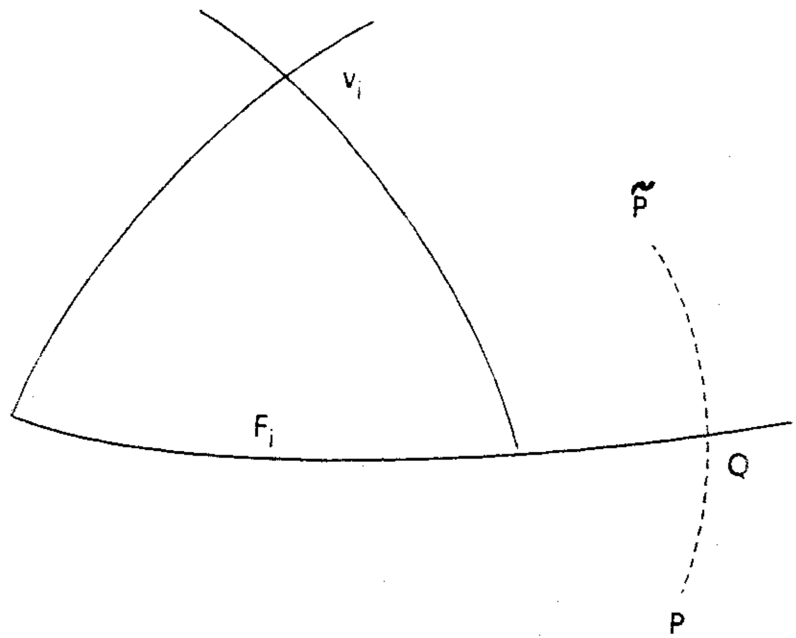

Figure 15

THEOREM 21: If all edges of V are obtuse, the interior of V^* is its set of CP points.

PROOF: By classic result 3 if F is any face of V, the projection of V^* on F is F^* which by Theorem 20 is interior to F. Thus all points of V^* are CP points. □

THEOREM 22: If all dihedral angles of V are acute then all points interior to V are its CP points.

PROOF: It is obvious that each v_i projects internally on its opposite facet F_i and from this it follows at once that any point of V projects internally on all facets. Now if H is the projection of v_i on F_i, and T is the projection of H on F_j, we claim the great circle $\overline{V_iT}$ is orthogonal to F_{ji}, the facet of F_j opposite v_i.

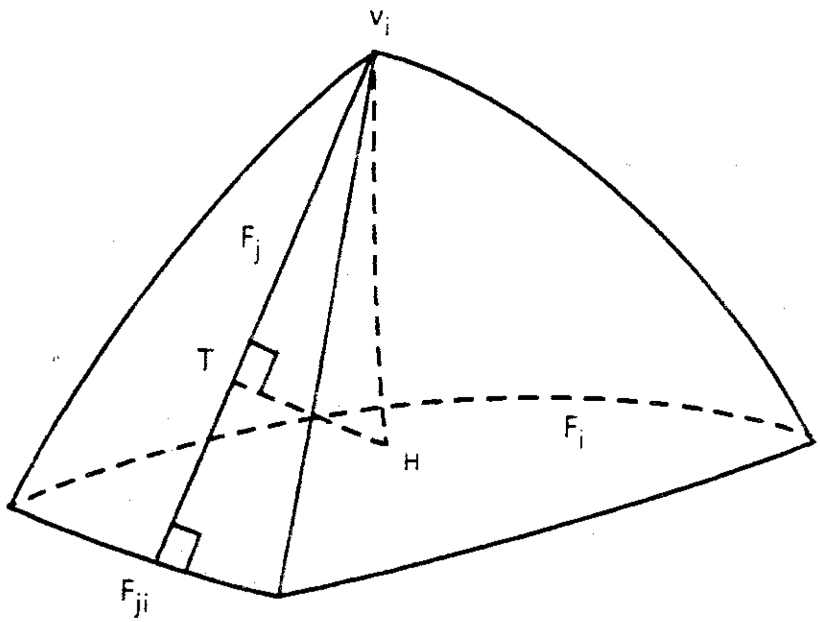

Figure 16

This may be seen by first noting that both $\overline{v_iH}$ and $\overline{TH}$ are orthogonal to F_{ji} and thus their spherical span is also. But $\overline{v_iT}$ is in this spherical span and so it too must be orthogonal to F_{ji}.

We have thus shown that each vertex of any facet of V projects internally on an opposite facet. By induction then every face of V has the same property.

Now if P is any interior point of V and F any face, let g be the projection of P on $\hat{F}$. Then F is in some facet G of V. The projection of P on $\hat{G}$ is interior to G call it h and the projection of h on $\hat{F}$ is g and by induction must be interior to F.

□

Other theorems in earlier sections can be given similar geometric proofs but these examples may suffice to illustrate the possibilities in this direction.

11. OPEN QUESTIONS

Given a rational nonsingular square matrix D and a rational point b in the interior of Pos (D), b has to satisfy property 2 of Section 3, to be a CP-point for Pos (D). Property 2 consists of 2^n-2 conditions. It is not known whether there is a polynomially bounded algorithm for checking property 2.

Identifying classes of matrices D for which Pos (D) can be shown to have a CP-point, which can be computed efficiently, has important applications in solving the corresponding LCP S efficiently (Section 4).

We have shown that the set of CP-points for the simplicial cone Pos (D) is an open polyhedral cone when it is nonempty (Theorem 4). This cone is defined by a system of inquality constraints consisting of a large number of constraints (2^n sets of constraints). However for n = 2, 3, we have proved that the set of CP-points is an open simplicial cone, and this result also held in all the examples that we examined for n = 4. It seems unlikely that the set of CP-points will be a simplicial cone in general, because the system of constraints defining it consists of a large number of constraints, but it would be interesting to settle the issue definitively.

12. A GENERALIZATION

In this section, we will discuss a generalization of the properties of being a CP-point or a PC-vector. Here, $R^{n \times n}$ denotes real square matrices of order n, which may or may not be nonsingular. We now state two properties 10, and 11.

10. The pair (D,q), where $D \in R^{n\times n}$ and $q \in R^n$, is said to have this property if for each nonempty $J \subset \Gamma$, the orthogonal projection of q onto the subspace $LH(D_{.J})$ lies in the relative interior of the cone Pos $(D_{.J})$.

11. The pair (M, q), where $M \in R^{n\times n}$ and $q \in R^n$, is said to have this property if for each nonempty $J \subset \Gamma$, the linear complementarity problem $(-q_J, M_{JJ})$ has a solution $(\tilde{w}_J, \tilde{z}_J)$ with $\tilde{z}_J > 0$.

Note that the definition of property 10 is minimal in the sense that every subset $J \subset \Gamma$ must be checked, as the following examples show. The listed D and q have the projection property for every subset of {1, 2, 3} except the ones listed:

$$D = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} 10 \\ 10 \\ -1 \end{pmatrix}, \qquad J = \{1, 2, 3\},$$

$$D = \begin{pmatrix} 0 & 1 & 4 \\ 1 & 5 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} 13 \\ 18 \\ 3 \end{pmatrix}, \qquad J = \{1, 2\}$$

$$D = \begin{pmatrix} 1 & -1 & -10 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ 100 \\ 1 \end{pmatrix}, \qquad J = \{1\}$$

Notice that in Property 10, D is not required to be nonsingular. Likewise in Property 11, M may not be nonsingular. We define

$P\,10_n = \{D \in R^{n\times n}$: there exists at least one $q \in R^n$ such that (D, q) satisfies property 10}.

$P\,11_n = \{M \in R^{n\times n}$: there exists at least one $q \in R^n$ such that (M, q) satisfies property 11}.

By our earlier results, if $D \in R^{n \times n}$ is nonsingular, $(D\ q)$ has property 10 iff $(D^T D, D^T q)$ has property 11.

CONVEXITY RESULTS

PROPOSITION 1: For $D \in R^{n \times n}$ the set of $q \in R^n$ such that (D, q) has property 10 is convex and open.

PROOF: Let (D, q) have property 10 and $\emptyset \neq U \subset \Gamma$. Let $J \subset U$ be a maximal subset such that $D_{.J}$ has independent columns. Then $LH(D_{.U}) = LH(D_{.J})$ and the projection of q onto $LH(D_{.U})$ is $D_{.J}\, \alpha^J$ where,

$$\alpha^J = D_{.J}^{\dagger}\, q\,,$$

where $D_{.J}^{\dagger} = ((D_{.J})^T D_{.J})^{-1} (D_{.J})^T$ is the pseudo-inverse of $D_{.J}$. The projection $D_{.J}\ \alpha^J$ is in the relative interior of $Pos(D_{.J})$ iff $\alpha^J > 0$. For $i \in U \setminus J$, there exist constants f_{ij} such that

$$D_{.i} + \sum_{j \in J} f_{ij} D_{.j} = 0\,,$$

from the definition of J. Choose $\epsilon_i > 0$, $i \in U \setminus J$, such that

$$\alpha_j^J + \sum_{i \in U \setminus J} \epsilon_i f_{ij} > 0\,,$$

for all $j \in J$ which is possible since $\alpha^J > 0$. Let α^U be a vector with components ϵ_i for $i \in U \setminus J$, and

$$\alpha_j^J + \sum_{i \in U \setminus J} \epsilon_i f_{ij}$$

for $j \in J$.

Then $\alpha^U > 0$ and

$$D_{.J}\ \alpha^J = D_{.U}\, \alpha^U\,.$$

Therefore the projection of q onto $LH(D_{.U})$ is given by $D_{.U}\alpha^U$, $\alpha^U > 0$.

By continuity, there is a neighborhood B^U of q such that $\tilde{a}^J = D_{\cdot J}{}^T\tilde{q} > 0$ and $\tilde{a}^U$ constructed as above satsifies $D_{\cdot J}\tilde{a}^J = D_{\cdot U}\tilde{a}^U$ and $\tilde{a}^U > 0$ for all $\tilde{q} \in B^U$. Then for any $p \in \cap_U B^U$, the projection of p onto $LH(D_{\cdot U})$ is given by $D_{\cdot U} a^U$ for $a^U > 0$, for each nonempty $U \subset \Gamma$. Therefore the set of q such that (D, q) has property 10 is open.

If (D, q^1) and (D, q^2) have property 10, then for fixed $U \subset \Gamma$, $U \neq \varnothing$, the projections of q^1 onto $LH(D_{\cdot U})$ and q^2 onto $LH(D_{\cdot U})$, are given by $D_{\cdot U}a^1$, $a^1 > 0$ and $D_{\cdot U}a^2$, $a^2 > 0$ respectively. Since projection is a linear operation, the projection of $(1-\lambda)q^1 + \lambda q^2$ onto $LH(D_{\cdot U})$ is $(1-\lambda)\ D_{\cdot U}a^1 + \lambda D_{\cdot U}a^2 = D_{\cdot U}\,((1-\lambda)a^1 + \lambda a^2)$, where $(1-\lambda)a^1 + \lambda a^2 > 0$ for $0 \leqq \lambda \leqq 1$. Since U was arbitrary, $(D, (1-\lambda)q^1 + \lambda q^2)$ has property 10 for $0 \leqq \lambda \leqq 1$, which proves the convexity.

□

Consider the following matrix:

$$D = \begin{pmatrix} -1 & 1 & 20 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that

$$D^{-1} = (1/3) \begin{pmatrix} -3 & 1 & 59 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

It can be verified that both D and $D^{-1} \notin P10n$. Also, this matrix D is nonsingular. Since $D \notin P10n$, this implies that $M = D^TD$ given below is $\notin P11n$.

$$M = D^TD = \begin{pmatrix} 1 & -1 & -20 \\ -1 & 10 & 23 \\ -20 & 23 & 402 \end{pmatrix}$$

PROPOSITION 2: For fixed $q \in R^n$, $n \geqq 2$, the set of matrices $M \in P_n \cap P11_n$ such that (M, q) has property 11 is not convex.

PROOF: Take

$$M = \left(\begin{array}{cc|c} 1 & -15 & 0 \\ 1 & 2 & \vdots \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right), \quad N = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -15 & 2 & \vdots \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right), \quad q = \begin{pmatrix} 2 \\ 3 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

M and N are P-matrices, (M, q) and (N, q) have property 11 but ((1/2) (M + N), q) does not have property 11.

□

CONJECTURE: For fixed $q \in R^n$, the set of PD symmetric $M \in R^{n \times n}$ such that (M, q) has property 11 is convex.

DEFINITION: A set $\Delta \subset R^n$ is *starlike* if there exists a point $c \in \Delta$ such that for any point $p \in \Delta$ the line segment $\overline{cp}$ also lies in Δ. The set of points $c \in \Delta$ with this property is called the kernel of Δ.

PROPOSITION 3: The set P_n is starlike with the identity matrix I in its kernel.

PROOF: Let $M \in P_n$ and $0 \leqq \alpha \leqq 1$. A characterization of P_n is that $M \in P_n$ if and only if the real eigenvalues of every principal submatrix of M are positive [3, 15]. If $\lambda_J > 0$ is an eigenvalue of a principal submatrix M_{JJ}, then $(1 - \alpha) + \alpha\lambda_J > 0$ is an eigenvalue of $[(1 - \alpha)I + \alpha M]_{JJ}$. Therefore the real eigenvalues of every principal submatrix of $(1 - \alpha) I + \alpha M$, for $0 \leqq \alpha \leqq 1$, are positive, and $(1 - \alpha) I + \alpha M \in P_n$ for $0 \leqq \alpha \leqq 1$.

□

Observe that the set P_n is not convex for $n \geqq 2$: $\begin{pmatrix} 1 & -15 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -15 & 2 \end{pmatrix}$ are P-matrices but $(1/2)\left(\begin{pmatrix} 1 & -15 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -15 & 2 \end{pmatrix}\right) = \begin{pmatrix} 1 & -7 \\ -7 & 2 \end{pmatrix}$ is not.

PROPOSITON 4: For fixed $q \in R^n$, $q > 0$, the set of matrices $M \in P_n$ such that (M, q) has property 11 is starlike with I in its kernel.

PROOF: Let (M, q) have property 11, where $M \in P_n$, $q \in R^n$, $q > 0$. It suffices to prove that $((1-\lambda) M + \lambda I, q)$ also has property 11, for $0 \leqq \lambda \leqq 1$. The proof of this is by induction on n. Since (M, q) and (I, q) have property 11, consider only $0 < \lambda < 1$. The result is trivial for $n = 1$. Assume it holds for P-matrices of order $< n$. Property 11 for (M, q) means that $M_{JJ}^{-1} q_J > 0$ for every subset $J, \varnothing \neq J \subset \Gamma$. Since property 11 is hereditary with respect to principal submatrices, the induction hypothesis yields that $[(1-\lambda) M_{JJ} + \lambda I_{JJ}]^{-1} q_J > 0$ for every subset J, $\varnothing \neq J \subset \Gamma$, of cardinality $|J| < n$. Thus all that remains is to prove $[(1-\lambda) M + \lambda I]^{-1} q > 0$.

The kth component of $z = [(1-\lambda) M + \lambda I]^{-1} q$ is, by Cramer's rule,

$$z_k = \frac{\det\left([(1-\lambda) M + \lambda I]_{\cdot\{1,\ldots,k-1\}} \quad q \quad [(1-\lambda) M + \lambda I]_{\cdot\{k+1,\ldots,n\}} \right)}{\det [(1-\lambda) M + \lambda I]} \tag{26}$$

Note that the denominator is positive since $(1-\lambda) M + \lambda I$ is also a P-matrix. The determinant is a multilinear form, and the numerator expands into a sum of determinants of the form det A, where $A_{\cdot i} = (1-\lambda) M_{\cdot i}$ or $A_{\cdot i} = \lambda I_{\cdot i}$ for $i \neq k$, and $A_{\cdot k} = q$. Let $J = \{j_1, \ldots, k, \ldots, j_r\}$ be the indices such that $A_{\cdot i} = \lambda I_{\cdot i}$ for $i \notin J$. Then

$$\det A = \lambda^{n-|J|} (1-\lambda)^{|J|-1} (M_{JJ}^{-1} q_J)_k \det M_{JJ} > 0$$

since $0 < \lambda < 1$, (M, q) has property 11, and M is a P-matrix. (The subscript k here refers to the element k in the index set $J = \{j_1, \ldots, k, \ldots, j_r\}$.) Therefore the numerator in (26) is positive, and so $[(1-\lambda) M + \lambda I]^{-1} q > 0$, which completes the induction step.

□

Note that Z_n is convex, but M_n is not: $\begin{pmatrix} 3 & -16 \\ -2 & 11 \end{pmatrix}$ and $\begin{pmatrix} 3 & -2 \\ -16 & 11 \end{pmatrix}$ are M-matrices

but $$(1/2)\left(\begin{pmatrix} 3 & -16 \\ -2 & 11 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -16 & 11 \end{pmatrix}\right) = \begin{pmatrix} 3 & -9 \\ -9 & 11 \end{pmatrix}$$ is not.

The following theorem from M. Fiedler and V. Ptak [3] will be useful.

THEOREM 23: Let $A, B \in Z_n$, all the real eigenvalues of A be positive, and $A \leqq B$.

Then 1) $A^{-1} \geqq B^{-1} \geqq 0$;

2) all the real eigenvalues of B are positive;

3) $\det B \geqq \det A > 0$.

PROPOSITION 5: The set M_n is starlike with the identity matrix I in its kernel.

PROOF: Let $M \in M_n = Z_n \cap P_n$. It is straightforward to verify that $(1-\lambda)I + \lambda M$ for $0 \leqq \lambda \leqq 1$ is in both Z_n and P_n.

□

PROPOSITION 6: For any $M \in M_n$ and $q > 0$, $q \in R^n$, the pair (M, q) has property 11.

PROOF: Let $M \in M_n$, $q \in R_n$, $q > 0$, and $\varnothing \neq J \subset \Gamma$. Then M_{JJ} is also an M-matrix, hence a Z-matrix with positive real eigenvalues. By Theorem 23 above, $M_{JJ}^{-1} \geqq (\mathrm{diag}\,(M_{JJ}))^{-1} \geqq 0$ is nonnegative with positive diagonal elements, and therefore $M_{JJ}^{-1} q_J > 0$. This proves property 11 is for the pair (M, q). □

Generalization of the class M_n.

M-matrices have many nice properties with respect to the linear complementarity problem. However, they have more structure than is really needed for linear complementarity proofs. We argue that $P11_n \cap P_n$ is the natural generalization of M_n, and that $P11_n \cap P_n$ contains the essence of the class M_n as far as linear complementarity is concerned.

Both M_n and $P11_n \cap P_n$ are nonconvex, both are starlike from I. When $M \in M_n$, for every $q > 0$, (M, q) has property 11; whereas for $M \in P11_n \cap P_n$, there exists a $q > 0$, such that (M, q) has property 11. When $M \in M_n$, for all $\varnothing = J \subset \Gamma$ and every $q > 0$, $(M_{JJ})^{-1} q_J > 0$; whereas, for $M \in P11_n \cap P_n$, there exists a $q > 0$ such that for all $\varnothing = J \subset \Gamma$, $(M_{JJ})^{-1} q_J > 0$. Similarly, when $M \in M_n$, the statement "for any $q \in R^n$, the LCP (q, M) can be solved with at most n complementary pivot steps via the parametric LCP $(q + \alpha p, M)$, is true for every $p > 0$" whereas, for $M \in P11_n \cap P_n$, there exists a $p > 0$ such that the above statement within quotes is true.

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