Magnetohydrodynamic Flow Between a Solid Rotating Disk and a Porous Stationary Disk

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MAGNETOHYDRODYNAMIC FLOW BETWEEN A SOLID ROTATING DISK
AND A POROUS STATIONARY DISK

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Abstract. In this paper we examine the flow of a conducting fluid between a solid rotating disk and a stationary porous disk with uniform injection of fluid through the porous disk in the presence of an axial magnetic field. The equations of motion are solved using least change secant update quasi-Newton and modern root finding techniques. The fluid motion depends on the cross-flow Reynolds number, rotational Reynolds number and Hartmann number. The effects of the parameters on the flow field are presented graphically.

1. INTRODUCTION

The Navier-Stokes equations for the steady incompressible flow between two infinite coaxial disks are amenable to similarity transformations. In the context of hydrodynamics, Batchelor [1] used similarity transformations to qualitatively study the solution for different ranges of Reynolds numbers $R_1$ and $R_2$ based upon the angular velocities of the disks and the gap length, respectively. Stewartson [2] obtained a series solution for small values of $R_1$ and $R_2$. Several later investigations [3]–[6] used analytical and numerical techniques. The flow between rotating disks with injection on the porous disk was thoroughly examined by Wang and Watson [7]. Several other references can be found in [7].

Srivasthava and Sharma [8] extended the solid rotating disks problem to MHD flow obtaining a solution for rotational Reynolds number $R_1 < 1$. This problem was analyzed analytically and numerically by Stephenson [9] for arbitrary Reynolds number $R$ and Hartmann number $M$. Chandrasekhar and Rudriah [10] obtained solutions for the two dimensional stationary case with small suction and injection velocities. Later Chandrasekhar and Rudriah [11] studied the case of axisymmetric flow between a rotating porous disk and a stationary porous disk with suction and

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injection, restricting the rotation and magnetic field parameters. Recently, Agarwal and Bhargava [12] studied numerically the flow between a solid rotating and a stationary porous disk with suction.

This paper examines the flow of a conducting fluid between a solid rotating and a stationary porous disk. At the stationary porous disk, fluid is injected with uniform velocity \( \omega \). By using suitable similarity transformations, the governing nonlinear partial differential equations are reduced to nonlinear ordinary differential equations. These equations are solved numerically using a least change secant update quasi-Newton and modern root finding methods. The problem depends on the nondimensional parameters \( R, R_1 \) and \( M \) which are the cross-flow Reynolds number, rotational Reynolds number and Hartmann number respectively. The effects of these parameters on the flow field are shown graphically.

2. FORMULATION OF THE PROBLEM

Consider the flow of an incompressible fluid of density \( \rho \), viscosity \( \mu \) and electrical conductivity \( \sigma \) bounded by two coaxial disks at \( z = 0 \) and \( z = h \) using a cylindrical polar coordinate system \((r, \theta, z)\). The \( z \)-axis is the axis of the disks. The upper disk at \( z = h \) is rotating at a constant angular velocity \( \Omega \). The lower stationary porous disk has fluid injected with a uniform velocity \( \omega \). A uniform magnetic field of strength \( B_0 \) is imposed in the \( z \) direction. In the analysis of this problem it is assumed that the distance \( h \) between the two disks is small compared to the radius, \( r_0 \), of the disks and that the edge effects are negligible.

The governing magnetohydrodynamic (MHD) equations of motion for the steady flow [9] are

\[
\rho \left( \dot{\vec{q}} \cdot \nabla \right) \vec{q} = -\nabla p + \mu \nabla^2 \vec{q} + \vec{J} \times \vec{B}, \quad (1)
\]
\[
\nabla \cdot \vec{q} = 0, \quad (2)
\]
\[
\nabla \cdot \vec{B} = 0, \quad (3)
\]
\[
\nabla \times \vec{B} = \mu_m \sigma(\vec{E} + \vec{q} \times \vec{B}), \quad (4)
\]
\[
\nabla \times \vec{E} = 0, \quad (5)
\]
\[
\vec{J} = \sigma(\vec{E} + \vec{q} \times \vec{B}), \quad (6)
\]
\[
\nabla \cdot \vec{J} = 0, \quad (7)
\]

where \( \nabla \) and \( \nabla^2 \) are the standard gradient and Laplacian operators expressed in cylindrical coordinates, and where \( \vec{q} = (u_r, u_\theta, u_z) \), \( \vec{J} = (J_r, J_\theta, J_z) \), \( \vec{E} = (0, 0, B_0) \), \( \vec{E} = (E_r, E_\theta, E_z) \), \( \mu_m \) and \( p \) are the velocity field, current density, magnetic field, electric field, magnetic permeability and pressure respectively. The components of the vectors refer to the \( r, \theta \) and \( z \) directions respectively.

For axisymmetric flow, equation (5) gives

\[
E_\theta = 0. \quad (8)
\]

Assume that the induced magnetic field is small compared to the imposed magnetic field. This gives

\[
\vec{B} = \dot{\xi} B_0.
\]
where \( \hat{z} \) is the unit vector in the direction of the positive z-axis.

Using these approximations, equation (6) gives

\[
\begin{align*}
J_\theta &= -\sigma B_0 u_r, \\
J_z &= \sigma E_z.
\end{align*}
\tag{9}
\]

Assuming

\[ E_r = -\chi B_0 \Omega r \tag{11} \]

yields

\[ J_r = \sigma B_0 (u_\theta - \chi \Omega r), \tag{12} \]

where \( \chi \) is a dimensionless quantity denoting the strength of the induced radial electric field and is equal to \( \omega_\text{ef} / \Omega \) for insulated disks. \( \omega_\text{ef} \) is the average velocity of the fluid.

Using the equations of motion (1) and (2), assuming the flow is axisymmetric, and incorporating equations (8), (9), (10) and (12) (see Stephenson [9]) yields

\[
\begin{align*}
\frac{\partial}{\partial r} (ru_r) + \frac{\partial}{\partial z} (ru_z) &= 0, \\
u_r \frac{\partial u_r}{\partial r} + u_s \frac{\partial u_r}{\partial z} - \frac{u_s^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \nabla^2 u_r - \frac{u_r}{r^2} \right) - \frac{\sigma B_0^2 u_r}{\rho}, \\
u_r \frac{\partial}{\partial r} (ru_\theta) + u_s \frac{\partial u_\theta}{\partial z} &= \nu \left( \nabla^2 u_\theta - \frac{u_\theta}{r^2} \right) - \frac{\sigma B_0^2 (u_\theta - \chi \Omega r)}{\rho}, \\
u_r \frac{\partial u_z}{\partial r} + u_s \frac{\partial u_z}{\partial z} &= -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 u_z,
\end{align*}
\tag{13-16} \]

where

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \]

and \( \nu = \mu / \rho \) is the kinematic viscosity. Writing \( u = u(r, z) \) and \( p = p(r, z) \), the boundary conditions are:

\[
\begin{align*}
u_r(r, 0) &= 0, \quad u_\theta(r, 0) = 0, \quad u_z(r, 0) = w, \\
u_z(r, h) &= 0, \quad u_\theta(r, h) = \Omega r, \quad u_z(r, h) = 0. \tag{17}
\end{align*}
\]

Using the following similarity transformations:

\[
\begin{align*}
\eta &= \frac{z}{h}, \quad U_r = \frac{w r f'(\eta)}{2 h}, \quad U_\theta = \Omega r g(\eta), \quad U_z = -w f(\eta), \\
p &= -r^2 A w^2 + P(\eta),
\end{align*}
\]

equations (13)-(16) reduce to

\[
\begin{align*}
f''' &= R((f')^2/2 - f f'' - \lambda g^2 - A) + M^2 f', \tag{18} \\
g'' &= R(f' g - f g') + M^2 (g - \chi), \tag{19} \\
P &= \rho \left( \frac{w^2}{2} - \frac{w^2}{2} f^2 - \frac{v w}{h} f' \right) + P_0.
\end{align*}
\]
The constant $F_0$ is the pressure at the porous stationary disk, $A$ is a constant of integration,

$$R_1 = \frac{\Omega h^2}{\nu}, \quad R = \frac{wh}{\nu}, \quad M = B_0 h \left( \frac{\sigma}{\rho \nu} \right)^{1/2}, \quad \lambda = \frac{2 R^2}{R^2},$$

and

$$\chi = \int_0^1 g(\eta) d\eta. \quad (20)$$

The boundary conditions (17) reduce to

$$f(0) = -1, \quad f'(0) = 0, \quad g(0) = 0, \quad (21)$$

$$f(1) = 0, \quad f'(1) = 0, \quad g(1) = 1. \quad (22)$$

Differentiating equation (18) with respect to $\eta$ to remove the constant $A$ gives

$$f^{(4)} = -R(f^{(m)} + 2\lambda gg') + M^2 f'' \quad (23)$$

3. LIFT AND TORQUE

Letting the pressure at the edge of the disk (at $r = r_0$) be $p_e$, the similarity transformation for $p$ yields

$$p - p_e = -\rho (r^2 - r_0^2) \frac{Au^2}{4h^2}.$$ 

The lift force is

$$L = 2\pi \int_0^{r_0} (p - p_e) r dr = \frac{\pi \rho w^2 r_0^4 A}{8h^2},$$

the shear stress is

$$\tau_{x\theta} = \mu \frac{\partial u_\theta}{\partial z} = \frac{\mu \Omega}{h} rg'(\eta),$$

and the torque is

$$T = 2\pi \int_0^{r_0} \tau_{x\theta} r^2 dr = \frac{\pi \rho \nu \Omega g'(\eta)}{2h} r_0^4.$$

4. NUMERICAL METHOD

Following the format in Heruska [13], define

$$\tilde{X} = \begin{pmatrix} f''(0) \\ f'''(0) \\ g'(0) \end{pmatrix}.$$ 

Let $f(\eta; \tilde{X}), g(\eta; \tilde{X})$ be the solution of the initial value problem given by (19) and (23) with initial conditions (21) and (24), where $\chi$ is treated as a constant. The original two-point boundary value problem described by equations (19) and (21)-(23) is numerically equivalent to solving the nonlinear system of equations

$$\hat{F}(\tilde{X}) = \begin{pmatrix} f(1; \tilde{X}) \\ f'(1; \tilde{X}) \\ g(1; \tilde{X}) - 1 \end{pmatrix} = 0 \quad (25)$$
together with satisfying (20). Solving (25) amounts to shooting from \( \eta = 0 \) to \( \eta = 1 \), until an appropriate value of \( \bar{X} \) is determined such that the boundary conditions at \( \eta = 1 \) are met. However, shooting from \( \eta = 0 \) to \( \eta = 1 \) proved unsuccessful on this problem, due to the initial slope of one of the functions \( (f') \) being very large. Next, a multiple shooting technique was tried. For this define

\[
X = \begin{pmatrix}
    f(.5) \\
    f'(.5) \\
    f''(.5) \\
    f'''(.5) \\
    g(.5) \\
    g'(.5)
\end{pmatrix}
\]

(26)

Then the boundary conditions (21) and (22) give the following nonlinear system of equations

\[
F(X) = \begin{pmatrix}
    f(0; X) + 1 \\
    f'(0; X) \\
    g(0; X) \\
    f(1; X) \\
    f'(1; X) \\
    g(1; X) - 1
\end{pmatrix} = 0.
\]

(27)

Define

\[
Y(\eta) = (f(\eta), f'(\eta), f''(\eta), f'''(\eta), g(\eta), g'(\eta)),
\]

so that the value of \( F(X) \) is determined from \( Y(\eta) \) at \( \eta = 0 \) and \( \eta = 1 \). The vector \( Y(\eta) \) can be calculated from the first order system of ordinary differential equations

\[
\begin{align*}
Y_1' &= Y_2, \\
Y_2' &= Y_3, \\
Y_3' &= Y_4, \\
Y_4' &= M^2 Y_5 - R(Y_1 Y_4 + 2 \lambda Y_2 Y_6), \\
Y_5' &= Y_6, \\
Y_6' &= M^2 (Y_5 - \chi) - R(Y_1 Y_6 - Y_2 Y_5),
\end{align*}
\]

(28)

with initial conditions \( Y(.5) = X \), by integrating both forward and backward. An unusual aspect of this problem is the constant \( \chi \) in (20). This constant's value depends on the solution, specifically, the function \( g(\eta) \). In order to satisfy (20), \( g \) must be viewed as a function of \( X \) and \( \chi \) as well as \( \eta \). Thus the problem is to find the value of \( \chi \) such that

\[
\Psi(\chi) = \int_0^1 g(\eta; X, \chi) \, d\eta - \chi = 0
\]

(29)

and (27) is also satisfied. To solve this problem, a least change secant update quasi-Newton method was utilized for the nonlinear system part (27) and a hybrid secant and bisection technique was utilized for the root finding part (29). The quasi-Newton code used was HYBRD from the MINPACK subroutine package from Argonne National Laboratory [14]. The root finding code used was ROOT from the ODE subroutine package in Shampine and Gordon [15]. Using an initial guess for \( \chi \), HYBRD determined an \( X \) satisfying (27). The value of \( \Psi(\chi) \) using this \( X \) typically would not satisfy (29). The root finding routine used this value of \( \Psi(\chi) \) to predict the root \( \chi \). This iteration was continued until (29) was satisfied.
<table>
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<th>$g'(0)$</th>
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**Table 1.** Effects of $M^2$, $R$ and $R_1$ on $\chi$, $g'(0)$, $g'(1)$ and $A$.

5. DISCUSSION

Figure 1 shows the effect of the Hartmann number ($M$) on the axial, radial and tangential velocities for a fixed $R$ and $R_1$. From Figure 1a, the axial velocity $f'$ is always upward. The axial velocity decreases with increased $M$ up to just past the midplane, but increases with increased $M$ in the upper part of the plane. The radial velocity $f'$ is shown in Figure 1b. The radial velocity has a maximum value near the midplane of the disks for $M^2 < 64$. For larger values of $M^2$, the radial velocity reaches a maximum nearer the lower disk, levels off for a distance and falls to zero at the upper disk. The tangential velocity $g$ (Figure 1c) increases as $M^2$ increases for $\eta < .6$ but decreases in the area closer to the upper disk as $M^2$ increases.

Figure 2 shows the effect of $R_1$ on the axial, radial and tangential velocities for a fixed $M$ and $R$. Again the axial velocity is always upward (Figure 2a). Increasing $R_1$ has the effect of increasing the axial velocity. The effect of increasing $R_1$ on radial velocity (Figure 2b) is to move the maximum value towards the upper disk. Also, the maximum value is increased slightly. The effect of increasing $R_1$ on tangential velocity (Figure 2c) is to slightly decrease it.

Figure 3 shows the effect of $R$ on the axial, radial and tangential velocities for a fixed $M$ and $R_1$. Increasing $R$ also increases the axial velocity (Figure 3a), but with a less pronounced effect than increasing $R_1$. There is less effect on radial velocity (Figure 3b) for large values of $R$ than $R_1$. However, changes in small values of $R$ have more effect on radial velocity than changes in small values of $R_1$. The tangential velocity (Figure 3c) is effected by changes in $R$ significantly, unlike changes in $R_1$. The effect of increasing $R$ is to decrease tangential velocity.

Table 1 shows the effects of changing $M^2$, $R$ and $R_1$ on the induced radial electric field ($\chi$), the shear stress and torque at each disk ($g'(0)$ and $g'(1)$), and the lift force and edge pressure ($A$). Increasing $M^2$ increases $\chi$, $g'(0)$, $g'(1)$ and $A$. Increasing the rotational Reynolds number ($R_1$) decreases the electric field, shear stress and torque at the lower disk, as well as decreasing the lift force and edge pressure; however, the shear stress and torque at the upper disk are increased. The effect of increasing the cross-flow Reynolds number $R$ is, generally, the same as increasing $R_1$. However, the changes due to $R$ are more dramatic for $\chi$, $g'(0)$ and $g'(1)$ than the changes due to $R_1$. The reverse is true for $A$.  

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References


Figure Captions.

Figure 1. Effect of $M^2$ on flow field with $R = 2, R_1 = .447$ and $M^2 = 0, 16, 64, 256, 512$ (short dash, dotted, solid, long dash, dash dot). (a) Axial velocity $f(\zeta)$. (b) Radial velocity $f'(\zeta)$. (c) Tangential velocity $g(\zeta)$.

Figure 2. Effect of $R_1$ on flow field with $M^2 = 16, R = 2$ and $R_1 = 2, 4, 10, 16$ (short dash, dotted, solid, long dash). (a) Axial velocity $f(\zeta)$. (b) Radial velocity $f'(\zeta)$. (c) Tangential velocity $g(\zeta)$.

Figure 3. Effect of $R$ on flow field with $M^2 = 16, R_1 = 2$ and $R = 2, 4, 10, 16$ (short dash, dotted, solid, long dash) (a) Axial velocity $f(\zeta)$. (b) Radial velocity $f'(\zeta)$. (c) Tangential velocity $g(\zeta)$. 

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