Two Point Constraint Approximation in Structural Optimization

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Abstract

The use of constraint approximations is recognized as a primary means of achieving computational efficiency in structural optimization. Existing approximation methods are based upon the value of the constraint function and its derivatives at a single point. The present paper explores the use of approximations based upon the value of the constraint and its derivative at two points. Several candidate approximations are suggested and tested for randomly generated rational constraint functions. Several of the approximations prove to be superior to the single point approximations.

INTRODUCTION

One of the major obstacles to the widespread use of automated structural design (structural optimization) procedures is the need for large computational resources to perform repeated analyses of the structure throughout the design process. In the mid-seventies Schmit and coworkers (Refs. 1, 2) proposed the use of approximate analysis techniques as a way of overcoming this obstacle. Over the past few years the use of approximations in structural optimization has become more and more common. The popularity of the idea was underscored in a recent international symposium on structural optimization (Ref. 3) where a large number of papers (e.g. Refs. 4-10) employed approximation techniques.

Some approximation techniques are global in nature. They seek to approximate structural behavior in the entire design space or a large part of it. This is done by employing simplified structural models (e.g. Ref. 11) by employing reduced basis techniques (e.g. Ref. 12) or by sampling the design space and using multiple regression techniques to curve-fit these points with a simple functional representation (e.g. Ref. 13). However, more popular are local approximation techniques which rely on the derivatives of the response function at a design point to obtain an approximation which is valid in the neighborhood of that design point. One reason for the popularity of such techniques is that the derivatives of the response are usually required for the direction seeking procedure in the optimization algorithm. Thus, the approximations can be obtained for a very little extra computational cost by using these derivatives.

One obvious local approximation based on first order derivatives is a first order Taylor series (i.e. a linear approximation). However, often the accuracy of such an approximation is acceptable only for relatively small

changes in the design parameters. In most structural design problems a better approximation may be obtained by using a linear expansion in the reciprocals of the variables (e.g. Refs. 14-16) and this approximation is now widely used. Other approximations which are in use are rational function approximations (Refs. 17, 18) and a convex "conservative" approximation (Refs. 19, 20) which is a hybrid linear/reciprocal approximation that is biased to overestimate the criticality of constraint functions. Most of these approximations utilize only first derivatives, however some work (e.g. Ref. 21) has been done with second derivatives.

All of the above techniques use approximations based on a single design point. As the structure is being resized new approximations are constructed at new design points. This practice is wasteful because the information from previous design points is discarded and is not used to improve later approximations. One reason for the current practice of using only single point approximations is the large number of variables. There is very little theoretical work on approximating a function of many variables to fit data at a small number of points. In fact, most of the work is on approximating a function of a small number of variables based on data at a large number of points. The present work considers approximations to a function of many variables based on the value of the function and its derivatives at two points. Several candidate approximations are explored and the results for the more successful candidates are presented.

APPROXIMATION METHODS

Several two-point approximation methods were developed and compared against the two most commonly used one-point approximations. Additionally, two three-point methods were derived and compared with the two-point

approximations. These one- and multi-point approximations are described below. The points where the data is given are denoted X_1, X_2 and X_3 with components x_{1i}, x_{2i}, x_{3i}, i=1,...,n. The function to be approximated is g(X), and the one-point approximations use its value and its derivatives at X_2. The two-point approximations use the value of the function and its derivatives at two points, and the three-point approximations use the data at three points.

1. Linear approximation

This is the first-order Taylor series expansion:

$$g_\ell(X) = g(X_2) + \sum_{i=1}^{n} (x_i - x_{2i}) \frac{\partial g}{\partial x_i} (X_2) \tag{1}$$

2. Reciprocal approximation

This approximation is the first order Taylor series expansion in the reciprocals of the variables $y_i = 1/x_i$, i=1,...,n. Written in terms of the original variables, x_i,

$$g_r(X) = g(X_2) + \sum_{i=1}^{n} (x_i - x_{2i})(x_{2i}/x_i) \frac{\partial g}{\partial x_i} (X_2) \tag{2}$$

3. Modified reciprocal approximation

The modified reciprocal approximation (Ref. 17) is

$$g_m(X) = g(X_2) + \sum_{i=1}^{n} (x_i - x_{2i}) \frac{(x_{mi} + x_{2i})}{(x_{mi} + x_i)} \frac{\partial g}{\partial x_i} (X_2) \tag{3}$$

Note that for $x_{mi} = 0$ we get the reciprocal approximation, and for $x_{mi} \to \infty$ we get the linear approximation. The values of x_{mi} are assumed to be selected on the basis of experience in Ref. 17, but now they can be selected

to match the derivatives at X_1 (note that the value of the function and its derivatives at X_2 are matched by g_m for any values of x_{mi}). That is

$$\frac{\partial g_m}{\partial x_i}(X_1) = \frac{\partial g}{\partial x_i}(X_1) = \left(\frac{x_{mi} + x_{2i}}{x_{mi} + x_{1i}}\right)^2 \frac{\partial g}{\partial x_i}(X_2) \tag{4}$$

or

$$x_{mi} = \frac{x_{2i} - \eta_i x_{1i}}{(\eta_i - 1)} \tag{5}$$

where

$$\eta_i^2 = \frac{\partial g}{\partial x_i}(X_1) \Big/ \frac{\partial g}{\partial x_i}(X_2) \tag{6}$$

When the ratio of the derivatives is negative, it is an indication that the derivatives at X_1 cannot be matched. In that case x_{mi} is set to a very large number, so that the linear approximation is used for the i-th variable.

4. Quasi-Newton approximation

The quasi-Newton approximation is written as

$$g_q(X) = g(X_k) + \sum_{i=1}^{n} (x_i - x_{ki}) \frac{\partial g}{\partial x_i}(X_k) + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{kij}(x_i - x_{ki})(x_j - x_{kj}) \tag{7}$$

where for a two-point approximation k=2 and for a three-point approximation k=3. The matrix A_k is the quasi-Newton approximation to the Hessian matrix of g. Three popular update formulae for A_k were tried.

(i) Inverse Broyden-Fletcher-Goldfarb-Shanno (BFGS) update

$$A_{k+1} = A_k - \frac{A_k P_k P_k^T A_k}{P_k^T A_k P_k} + \frac{Y_k Y_k^T}{Y_k^T P_k} \tag{8}$$

where

$$Y_k = \nabla g(X_{k+1}) - \nabla g(X_k)$$

$$P_k = X_{k+1} - X_k$$

$$A_o = 0 \tag{9}$$

Note that unlike usage in optimization there is no need for a positive definite A, and $A_o = 0$ is a logical choice in the absence of additional knowledge.

(ii) BFGS update

$$A_{k+1} = \left(I - \frac{Y_k P_k^T}{Y_k^T P_k}\right) A_k \left(I - \frac{P_k Y_k^T}{Y_k^T P_k}\right) + \frac{Y_k Y_k^T}{Y_k^T P_k} \tag{10}$$

(iii) Davidon-Fletcher-Powell (DFP) update

$$A_{k+1} = \frac{(Y_k - A_k P_k)(Y_k - A_k P_k)^T}{(Y_k - A_k P_k)^T P_k} + A_k \tag{11}$$

Only the DFP update gave reasonable results.

5. Two-point projection method

The method is based on a projection of the point X on the line connecting X_1 and X_2 (see Fig. 1). The function g(X) is first approximated by a cubic Hermite polynomial at the projection point, P, and then linearly extrapolated to X. The projection point P is given as

$$P = X_1 + \xi(X_2 - X_1) \tag{12}$$

where

$$\xi = (X - X_1)^T(X_2 - X_1)/||X_2 - X_1||^2 \tag{13}$$

The quality of the approximation was found to be good only for interpolation (that is $0 \leq \xi \leq 1$) so that if ξ from Eq. (13) was larger than one or

smaller than zero it was set to the respective limit of zero or one. The approximation at P is then written as

$$g_p(X) = g_p(P) + \sum_{i=1}^{n} [(1 - \xi) \frac{\partial g}{\partial x_i} (X_1) + \xi \frac{\partial g}{\partial x_i} (X_2)](x_i - p_i) \qquad (14)$$

where

$$g_p(P) = N_1(\xi)g(X_1) + N_2(\xi)d_1 + N_3(\xi)g(X_2) + N_4(\xi)d_2 \qquad (15)$$

the $N_i(\xi)$ are the Hermite interpolation polynomials

$$N_1(\xi) = 1 - 3\xi^2 + 2\xi^3$$

$$N_2(\xi) = \xi - 2\xi^2 + \xi^3$$

$$N_3(\xi) = 3\xi^2 - 2\xi^3$$

$$N_4(\xi) = -\xi^2 + \xi^3 \qquad (16)$$

and

$$d_i = (X_2 - X_1)^T \nabla g(X_i) \qquad i = 1,2 \qquad (17)$$

6. Three-point projection method

The method is based on a projection of the point X onto the plane containing the points X_1, X_2 and X_3 where the data is given (see Fig. 2). The function g(X) is first approximated by a quadratic polynomial at the projection point, P, and then linearly extrapolated to X. A local coordinate system (s,t) is defined in the plane (see Fig. 2).

The computation begins by calculating two unit vector e_s and e_t in the plane and the calculation of the derivatives $\partial g/\partial s$ and $\partial g/\partial t$ at points X_1, X_2 and X_3. In the plane the function is approximated as

$$g_p(s,t) = g(X_3) + \frac{\partial g}{\partial s} (s - s_3) + \frac{\partial g}{\partial t} (t - t_3)$$

$$+ \frac{1}{2} a_{11}(s - s_3)^2 + a_{12}(s - s_3)(t - t_3) + \frac{1}{2} a_{22}(t - t_3)^2 \qquad (18)$$

and the three unknown coefficients are found from a least-square fit using the value of the function and its derivatives at points X_1 and X_2.

Once Eq. (18) is used to approximate g(P), the gradient vector $\nabla g(P)$ is found by linear interpolation in the s,t plane using $\nabla g(X_1)$, $\nabla g(X_2)$ and $\nabla g(X_3)$. Finally, g(X) is approximated as

$$g_p(X) = g_p(P) + (X - P)^T \nabla g(P) \qquad (19)$$

7. Exponential approximation

The exponential approximation is based on dividing the variables into two groups: those that have constant sign derivatives and those that alternate in sign. That is

$$j \varepsilon J_1 \text{ if } \frac{\partial g}{\partial x_j}(X_1) \frac{\partial g}{\partial x_j}(X_2) \geq 0$$

$$j \varepsilon J_2 \text{ if } \frac{\partial g}{\partial x_j}(X_1) \frac{\partial g}{\partial x_j}(X_2) < 0 \qquad (20)$$

A general function which replicates the behavior of the derivatives is

$$g_e(X) = c \exp - \left[\sum_{j \varepsilon J_1} \alpha_j x_j^{\beta_j} + \sum_{j \varepsilon J_2} \gamma_j (x_j - \delta_j)^2\right] \qquad (21)$$

The coefficients α_j, β_j, γ_j and δ_j are calculated to match the derivatives at X_1 and X_2. For $j \varepsilon J_1$ we get two equations

$$\frac{\partial g}{\partial x_j}(X_1) = \alpha_j \beta_j (x_{1j})^{\beta_j - 1} g(X_1)$$

$$\frac{\partial g}{\partial x_j}(X_2) = \alpha_j \beta_j (x_{2j})^{\beta_j - 1} g(X_2) \qquad (22)$$

which yields

$$\beta_j = 1 + \frac{\ell n[\frac{\partial g}{\partial x_j}(X_1)\, g(X_2) / \frac{\partial g}{\partial x_j}(X_2)\, g(X_1)]}{\ell n(x_{1j}/x_{2j})} \tag{23}$$

and

$$\alpha_j = - \frac{\partial g}{\partial x_j}(X_1) \beta_j x_{1j}^{\beta_j - 1} / g(X_1) \tag{24}$$

similarly, for $j \varepsilon J_2$, the equations

$$\frac{\partial g}{\partial x_j}(X_1) = -2g(X_1)\gamma_j(x_{1j} - \delta_j)$$

$$\frac{\partial g}{\partial x_j}(X_2) = -2g(X_2)\gamma_j(x_{2j} - \delta_j) \tag{25}$$

are solved to yield

$$\delta_j = \frac{x_{1j} g(X_1) \frac{\partial g}{\partial x}(X_2) - x_{2j} g(X_2) \frac{\partial g}{\partial x}(X_1)}{g(X_1) \frac{\partial g}{\partial x}(X_2) - g(X_2) \frac{\partial g}{\partial x}(X_1)} \tag{26}$$

$$\gamma_j = \frac{- \frac{\partial g}{\partial x}(X_1)}{2g(X_1)(x_{1j} - \delta_j)} \tag{27}$$

Finally, the constant c is calculated to match the function value at X_2.

$$c = \hat{g}/g(X_2) \tag{28}$$

where

$$\hat{g} = \exp - [\sum_{j \varepsilon J_1} \alpha_j x_{2j}^{\beta_j} + \sum_{j \varepsilon J_2} \gamma_j (x_{2j} - \delta_j)^2] \tag{29}$$

TEST PROCEDURE

Test functions

The displacements of a linear elastic structure may be found by solving a system of linear equations generated by a finite element model. The elements of the matrix of the equations (the stiffness matrix) are typically polynomials of the design variables. Therefore the displacements are rational functions of the design variables. The test functions selected for evaluating the various approximation are, therefore, chosen to be simple rational functions with random coefficients. In choosing the form of the rational function two properties of displacement functions were considered. First, for most structural optimization problems the displacements are finite for positive values of the design parameters. Second, when the design parameters are cross-sectional areas of bars or thicknesses of membrane elements, a simple scaling relation holds. When all the design parameters are multiplied by a factor, the resulting displacements and stresses are reduced by the same factor.

To achieve the above characteristics the test function g(X) is chosen to be in the form

$$g(X) = \frac{a(X)}{b(X)} \tag{30}$$

where

$$a(X) = \sum_{j=1} s_{1j} x_j \tag{31}$$

and

$$b(X) = \sum_{j=1}^{n} (s_{2j} x_j^2 + s_{3j} x_j) + \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} w_{jk} x_j x_k \tag{32}$$

The coefficients s_{1j}, s_{2j}, s_{3j} and w_{jk} are randomly generated using a two stage process. First, a random number is selected for each coefficient to

determine whether it is zero or not. The probability of being nonzero is denoted as p_1, p_2, p_3 and p_{12} for s_{1j}, s_{2j}, s_{3j} and w_{jk}, respectively. Second, if the coefficient is to be nonzero its value is selected by a random variable uniformly distributed in an interval (o,d_i), $i=1,\ldots,4$, where $i=1,2,3$ correspond to s_{ij} and $i=4$ to w_{jk}. Note that for any choice of the random coefficients $g(X)$ has the desired scaling properties.

Test points

Beside test functions it is also necessary to select sequence of design points. Two procedures are used for this purpose. The first procedure is completely random with each component x_i selected as a uniformly distributed random number in the interval (o,d). The second procedure seeks to imitate the sequence of points generated by an optimization procedure. It assumes that the design proceeds in a certain direction with random oscillations about that direction. This procedure follows five steps, requiring the generation of variables. In all cases below, each random variable has a uniform distribution in the indicated interval. The steps of the procedure are as follows:

1. An initial point X_1 is selected with x_{1j}, $j=1,\ldots,n$ a random number in $[0,1]$.

2. A direction vector U is chosen with each component a random number in $[0,1]$ and then U is normalized so that $||U|| = 1$.

3. A sequence of $m-1$ random numbers λ_i, $i=1,\ldots,m-1$ each in $[o,d]$ is generated. It is used to define m points X_{pi} moving along the given direction

$$X_{p1} = X_1$$

$$X_{pi} = X_{p(i-1)} + \lambda_{i-1} U \qquad i=2,\ldots,m \qquad (33)$$

4. Random perturbations normal to the direction of motion are generated by generating another random direction V_k, k=1,...,m-1 as in step 2, making it orthogonal to U and normalizing its length to 1.

5. The final design vector is obtained by adding V_k to X_{pk} as

$$X_k = X_{pk} + \alpha_k V_k \qquad k=2,\ldots,m \qquad (34)$$

where α_k is a random number in [o,r].

The m test points which are generated by either procedure were used to generate m-2 approximations. The value of g(X) at X_k, k=3,...,m is approximated by the three point approximations based on the function and its derivatives at X_{k-1}, X_{k-2} and X_{k-3}. Two-point approximations used data at X_{k-1} and X_{k-2}, and single-point approximations utilized the data at X_{k-1}.

RESULTS AND DISCUSSION

One- and two-point approximations

The constants used for the random generation of the function and the test points are given in Table 1. Ninety test points were generated by each procedure and eighty-eight approximations were generated. These are compared on the basis of average relative error, maximum relative error and the number of times (out of 88) that a given approximation was the best of all approximations.

The single and two-point approximation were first compared for the completely random test points. Table 2 presents the results for the linear, reciprocal, modified, quasi-Newton (DFP update) and projection approximations as a function of the number of design variables. The

projection 2-point approximation had the best performance with an average error 25-40% lower than that of the linear approximation which was the next best. The other approximations had very large maximum errors which indicated that their poor performance may be due to a small number of points with large errors. Therefore it was decided to check how much the performance will be affected if the permissible deviation from the linear approximation was limited.

The approximations were limited to 30% deviation from the linear approximation. That is

$$0.7 \leq \frac{|g_a(X) - g(X_2)|}{|g_\ell(X) - g(X_2)|} \leq 1.3 \tag{35}$$

where g_a represents g_r, g_m, g_q or g_p. The results with these corrected approximations are given in Table 3. It is seen that the quasi-Newton approximation is greatly improved and has average errors which are up to 12% lower than those of the linear approximation. The reciprocal and modified approximations are also greatly improved but they remain inferior to the linear approximation, and the projection approximation deteriorates due to the correction process.

Next, the single- and two-point approximations were compared for the directed set of test points. Because most of the design parameters vary monotonically the reciprocal approximation is much better with this sequence than the linear approximation. Therefore the projection method was modified to use reciprocal variables and the modified and quasi-Newton approximations were constrained with respect to reciprocal approximation (Eq. (35) with g_k replacing g_ℓ). Table 4 shows the results obtained with the directed

sequence of points. The reciprocal approximation was superior to all others with the quasi-Newton method close to it.

The superior performance of the reciprocal approximation for this case is not surprising in view of past studies (e.g. Refs. 14-16). Because it is exact when all variables are scaled up or down by the same factor, it tends to be accurate for this case of test points which are almost monotonically increasing for all variables. This case may mean that when the reciprocal approximation is much more accurate than the linear approximation, it is difficult to improve upon it using two point approximations.

The exponential approximation and the quasi-Newton approximations with the BFGS updates were generally inferior to the other approximations, and so results for these approximations are not presented.

Three-point approximations

The two-point and three-point projection and quasi-Newton methods performed well compared to the linear approximations. Therefore three-point versions of these methods were implemented and compared to the single- and two-point approximations. Based on previous experience only the quasi-Newton approximation was corrected by Eq. (35). The results for the completely random set of test points are summarized in Table 5. The three-point projection method was substantially better than the two-point version, while the improvement of the quasi-Newton method was marginal.

The results for the directed set are presented in Table 6. For this set the three-point approximations are generally poorer than the two-point ones. These results confirm the trend observed for the two-point approximations of improvements over the single-point approximations only for the random set.

CONCLUDING REMARKS

Several two-point approximations have been derived for use in constraint function approximations. A rational function with random coefficients displaying some of the characteristics of displacement and stress constraint functions was used for evaluating the approximations. Two sequences of random test point sequences were used in the evaluation. The first sequence was completely random and the second had random perturbation superimposed on monotonically increasing variables. For the first sequence the linear approximation was superior to the reciprocal approximation and a two-point projection method was substantially better than either of these single-point approximations. For the second (directed) sequence the reciprocal approximation was much more accurate than the linear approximation and slightly more accurate than the two-point approximations. To check whether the approximations can be improved by using more points, two three-point approximations were tested. They showed improvement for the random set of test points but not for the directed set.

The results indicate that the projection method can be used to substantially improve the accuracy of the approximation compared to linear approximation. However, in cases where the reciprocal approximation is very accurate because of its scaling property it is very difficult to improve upon it.

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TABLE 1

Ranges used for random variable generation

d_i, i=1,...,4: 25., 60., 20., 30. for first generation scheme

: 10., 10., 10., 10. for second generation scheme

$P_i = 0.8$, i=1,2,3: $P_{12} = 0.8$

d = 8, r = 1.0

TABLE 2

Errors for single- and two-point approximations for random 90-point sequence (average relative error, maximum relative error, and number of times each approximation was the best one).

Number of Variables	linear	reciprocal	modified	quasi-Newton	projection
5	0.249	0.680	0.444	0.559	0.183
	1.863	344.3	4.197	18.15	1.561
	15	4	21	5	29
10	0.0935	2.656	0.221	0.285	0.0543
	0.333	44.53	3.533	9.067	0.232
	16	5	13	25	31
20	0.0465	3.121	2.574	0.118	0.0354
	0.262	74.41	167.1	1.980	0.0975
	19	2	10	26	33
40	0.0174	1.681	0.118	0.0280	0.0123
	0.0499	34.49	1.989	0.373	0.0665
	14	0	8	31	35

TABLE 3

Errors for corrected single- and two-point approximations for random 90-point sequence (average relative error, maximum relative error, and number of times each approximation was the best one).

Number of Variables	linear	reciprocal	modified	quasi-Newton	projection
5	0.249	0.289	0.244	0.219	0.208
	1.863	2.255	1.471	1.471	1.471
	9	19	17	29	33
10	0.0935	0.117	0.0943	0.0822	0.0726
	0.333	0.425	0.272	0.323	0.254
	16	9	15	33	36
20	0.0465	0.0637	0.0546	0.0465	0.0384
	0.262	0.158	0.459	0.459	0.0978
	15	11	17	26	32
40	0.0174	0.0273	0.0193	0.0171	0.0141
	0.0499	0.0518	0.0810	0.0815	0.0428
	11	8	19	30	31

TABLE 4

Errors for single- and two-point approximations for directed 90-point sequence (average relative error, maximum relative error, and number of times each approximation was the best one).

Number of Variables	linear	reciprocal	modified	quasi-Newton	projection
5	0.0139	0.00585	0.00879	0.00614	0.00748
	0.741	0.143	0.375	0.233	0.162
	11	30	18	31	8
10	0.00554	0.00119	0.00308	0.00120	0.00121
	0.223	0.00837	0.0737	0.00837	0.00837
	13	12	38	11	22
20	0.00379	0.00106	0.00123	0.00112	0.00231
	0.0967	0.0141	0.0250	0.0141	0.0525
	4	16	50	8	11
40	0.00461	0.00232	0.00289	0.00245	0.00590
	0.0860	0.0220	0.0776	0.0234	0.269
	2	18	55	10	9

TABLE 5

Errors for single-, two- and three-point approximations for random 90-point sequence (average relative error, maximum relative error, number of times each approximation is the best one).

Number of Variables	linear	2-point projection	3-point projection	2-point quasi-Newton (corrected)	3-point quasi-Newton (corrected)
5	0.243	0.184	0.150	0.215	0.214
	0.183	1.560	1.545	1.471	1.471
	9	22	37	12	16
10	0.0903	0.0543	0.0438	0.0827	0.0811
	0.332	0.231	0.182	0.323	0.345
	8	23	38	6	16
20	0.0467	0.0356	0.0314	0.0468	0.0422
	0.262	0.0974	0.0742	0.459	0.159
	15	18	29	11	16
40	0.0175	0.0124	0.0114	0.0172	0.0180
	0.0499	0.0665	0.0557	0.0815	0.0815
	9	16	32	14	19

TABLE 6

Errors for single-, two- and three-point approximations for directed 90-point sequence (average relative error, maximum relative error, number of times each approximation is the best one).

Number of Variables	reciprocal	2-point projection	3-point projection	2-point quasi-Newton (corrected)	3-point quasi-Newton (corrected)
5	0.00585	0.00748	0.0144	0.00614	0.00786
	0.143	0.162	0.439	0.233	0.338
	26	9	14	39	30
10	0.00119	0.00121	0.00521	0.00120	0.00122
	0.00837	0.00837	0.102	0.00837	0.00749
	15	34	31	13	19
20	0.00106	0.00231	0.00285	0.00112	0.00119
	0.0141	0.0525	0.0493	0.0141	0.0163
	22	12	36	15	25
40	0.00232	0.00590	0.00283	0.00245	0.00234
	0.022	0.269	0.0233	0.0234	0.0220
	24	9	27	23	22

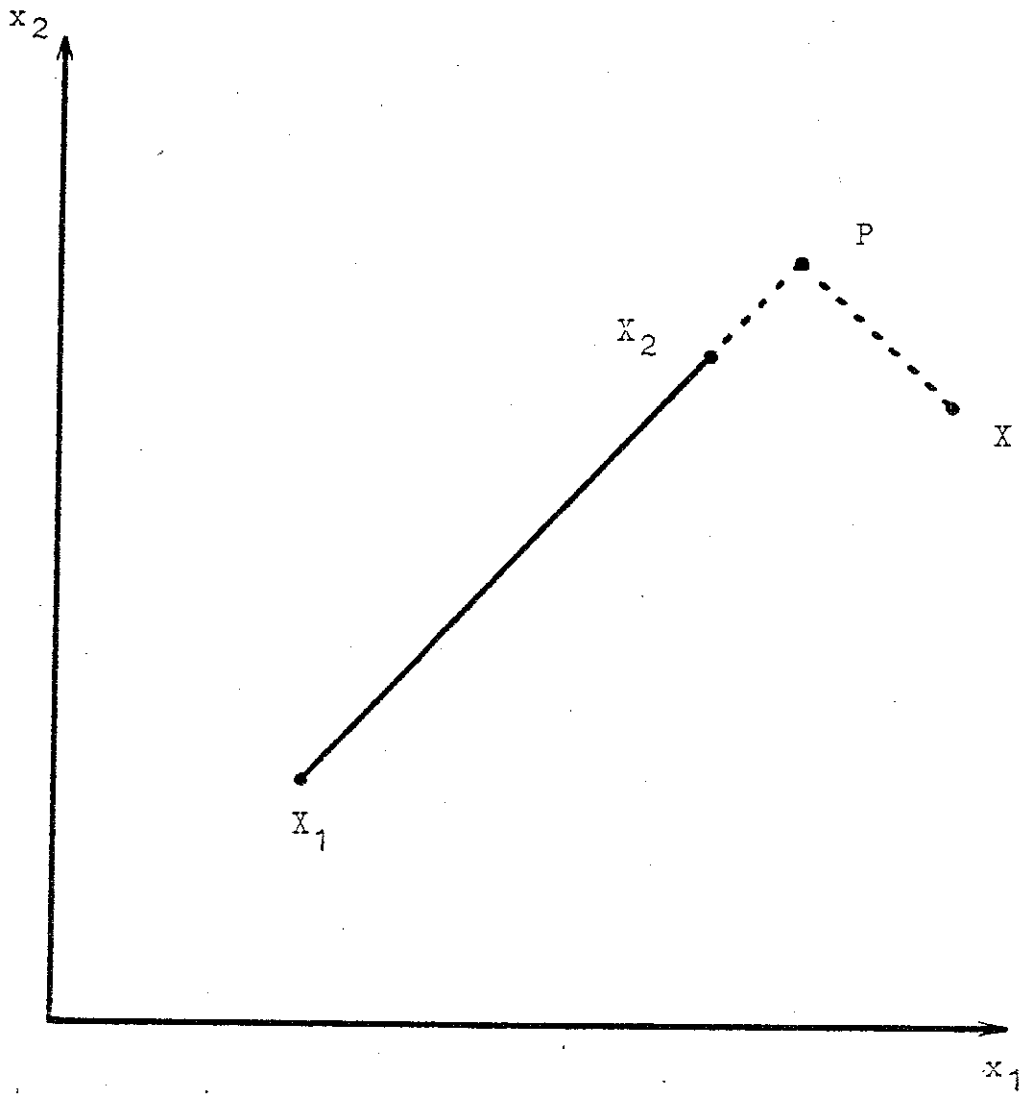

Figure 1. Two-Point Projection Method

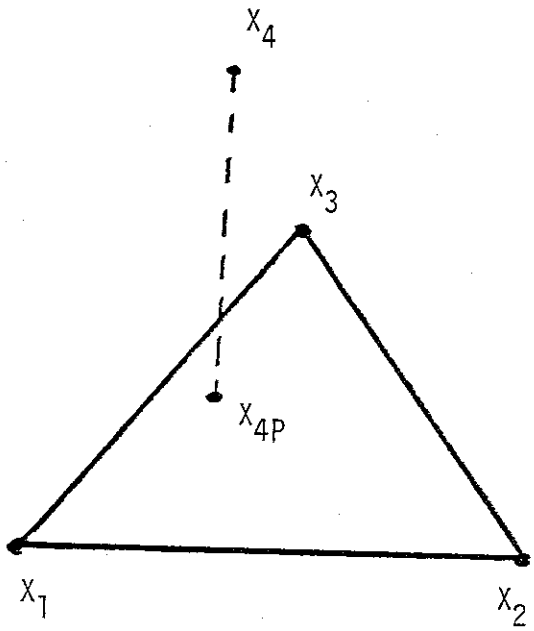

Figure 2. Three-Point Projection Method